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What has Cybernetics to do with Operational Research?*

STAFFORD BEER
The United Steel Companies Limited

Part 1 of this paper answers the question "What is cybernetics?" A brief historical review introduces a somewhat formal statement about the nature of systems and the way they are controlled. The unifying topic in the study of control in every context is an underlying identity of system: an example is taken from learning machines.

A comparably brief treatment of the question "What is operational research?" is given in Part 2. Operational research is thought to be the latest exemplification of scientific method rather than "a science". What is important about operational research is brought out through a description of an actual case study, and some of the activities which look like operational research but are not are mentioned. This leads to an attempted definition of operational research.

Part 3 tries to draw the answers to the first two questions together, and tc show how operational research and cybernetics are related. It is possible to pursue each activity in its own right; but it is also possible to practise operational research with essentially cybernetic models, and to study cybernetics by operational research techniques. This thought is generalized into the idea that cybernetics is the science of which operational research is the method.

Introduction

"THE Department of Operational Research and Cybernetics". This phrase is, I regret to say, rather a mouthful; and from what I hear some of my operational research colleagues find it too much to swallow. The last bit is not regarded as very respectable.

Now ten years ago operational research itself was not a particularly respectable activity. Today by contrast, the advertisement columns of the newspapers week by week attest to the fact that operational research is now highly respectable—and respected. It can only be a matter of time

* Paper presented to the Operational Research Society on 17 November, 1958.

before cybernetics too achieves the status which it must have: I am quite confident of this.

Now you will gather that my mood is not very apologetic. In fact this address is meant to be polemical. That is why I have given it the aggressive title that I have; that is why I am wearing a bullet-proof waistcoat.

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1. What is Cybernetics?

Cybernetics is the science of control; or, to expand it into Norbert Wiener's own words[1]: control and communication in the animal and the machine. Like most definitions, this one does not seem to say very much at first sight. To understand what cybernetics is about, one just must look at the origins of the work. As you probably know from Wiener's book, cybernetics originated in the work of a group of people representing a number of sciences. And the point I want to make, the first point, is that they did not get together to discuss the question of control in the first place. They had noted that the whole range of human thinking had been developing over the centuries into a pattern, a pattern represented today by a large number of subjects each generally discussed as if it were a valid object of study in its own right. Here was a pattern which represented the historical development of human thinking; could it be, however, that it might not represent the real world very accurately at all. Whole topics of enquiry might fall neatly between the stools of the established sciences.

The problem of control emerged from these early discussions as something which is common to every science. Soon it was discovered that the nature of this problem for scientists in every field was uniform in many respects. What I should like to call "the theory of organic control", which is applicable to mechanical and social systems as well as to biological, grew from this impetus. So the essence of cybernetics is to be inter-disciplinary.

Now it is very easy at this stage to talk about uniting the body of thinking, to speak of the unity of science and the indivisibility of knowledge. I am as aware as anyone of the pitfalls of this kind of speech. One can quickly throw oneself open to a charge of being some kind of alchemist, of having naïve and probably egotistical views about the complexity and wonder of nature and the amount of insight into it that one brain can possibly acquire and comprehend. But if you look on this concept of the unity of science simply in terms of the dimension of

control, which is the way the subject of cybernetics arose historically, such charges prove baseless. Cybernetics is not an attempt to comprehend the whole of science and knowledge in one glorious confusion, but rather an attempt to see whether the understanding of control problems so far acquired in many disparate sciences is being properly shared. Although a great deal is known about control in various departments of learning, little has been done about the nature of control *per se*. In attacking that problem, cybernetics must inevitably borrow the available information about its subject wherever that seems to be available; in this sense it is certainly a unifying influence on science at large.

Just as the concept of control appears in every science, any conclusions which can be reached about it must be applicable to many contexts. Very soon, Wiener and his associates realized that they were discussing something which was relevant to industrial, social and economic systems, as well as to the mechanical and biological ones. Now it is extremely difficult to talk scientifically about a topic which has so many utterly different origins, and also so many widely varied applications. For this reason, the essence of a developing science is that it has some formal technique for discussing the basic nature of its subject and of ridding itself of the trappings which make up the specific detail. Can we find a formal language in which to talk about problems of control wherever they arise?

I do not think there is any doubt about the answer: we can. In logic, in mathematics, in statistics, and in metamathematics, there are formal systems that we know how to manipulate. But in a general paper like this I must confine any formal analysis, not to some recondite algebra, but to a verbal version. Even in English, it is possible to discuss the nature of control formally, and without bringing in the details of many different sciences. But I must do this quickly, because it is rather a tiresome set of aphorisms that must be presented.

Control is the attribute of a system. A system is any collection of entities that can be understood as forming a coherent group. The fact of their being capable of being understood as a coherent group is precisely what differentiates a system from a meaningless collection or jumble of bits and pieces. Now that statement does not sound very important, but I think it is; because at once it reveals the relativity of this concept of a system. This means that we might have a series of numbers on the blackboard which to an uneducated man, or a savage, or indeed most of the general public, would appear to be a meaningless jumble of figures; but

to a mathematician this would be at once recognizable as a binomial series.

Therefore the property of being a system is as much a property of the observing system, which is I, or you, or whatever we use for the purpose as extensions of ourselves, as of the observed system itself.

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The structure of a system is its relatedness. A description of the way in which the system is interconnected *defines* its structure. The system and the structure are formal components of the analysis I wish to make, and are words which I shall henceforth use in this special way.

It seems to me that there are three main characteristics of a cybernetic system. The first is that it is extremely complex, to the point of indefinability. It may be that no real system is in principle indefinable; but many certainly are in practice, because (like the brain) they disintegrate once they are probed too closely.The reason for this characteristic of a cybernetic system is that the connectivity of its structure, of which I was speaking just now, is initially very great and changing continuously. Thus the brain, the economy, the industrial company; all these things are cybernetic systems under this first heading at the very least—the heading of extreme complexity.

May we note in passing that operational research itself often claims to deal with complex systems, and so it does. But when we talk about an operational research model we like to think, I suspect, that we are creating an isomorphic representation of our problem. If we use linear programming, for instance, to study a problem of the allocation of materials from five different sources to five different consumers, it may well be true that we can do this. But once we try and use it outside the context in which the straightforward matrix of small order can be constructed, and turn to game-theoretic analysis of the entire operations of a company, we are in very great difficulty. This is more than a practical problem, it is a methodological one; because we can no longer determine in any sense that our model is an isomorphism. To be quite formal: we must end up with a homomorphic representation—by which I mean, of course, that there is a many-one relationship from every segment of the situation being described to each element of the matrix which is describing it. This is a very severe methodological limitation on the analysis by operational research techniques of extremely complex systems. Cybernetics tries to answer the problem by the use of black Box theory. This is the theory which specifically begins by accepting the situation as indefinable in detail.

Now the second feature of a cybernetic system is its probabilism. Again, operational research deals with indeterminate systems. But we normally say this in the sense that *within* a rigid framework of related parts, most of the quantities which characterize the parts are not constants, they are distributed. For example, the relations which connect a queue to the machine which has to serve it are quite clear, distinct and unchanging. It is the waiting time and the service time that are the variables which lead us to call the system probabilistic. In cybernetics, however, the very structure of the system is indeterminate. The existence of connexion between entities which make up the system is itself problematic; the network of connectivity is itself defined by variables to which levels of probability can be assigned. If the levels fall below the threshold of statistical significance, then the system disintegrates and ceases to be recognizable as a system at all. If the level of threshold changes, the system itself changes, and is no longer describable as the system that it was.

The third major feature of cybernetic systems is the feature of self-regulation. How, after all, is a system to be controlled? In operational research, it is customary to design controls which can be operated within certain fiducial limits. This approach is a great advance over the less sophisticated notion that it is possible to have rigid error-free regulators. But consider the kind of system we have so far designated as cybernetic. It is to be so complex as to be indefinable; it is to be probabilistic not only in its operational parameters but in its very structure. It follows that no control at all designed to be imposed upon the system from outside could possibly work. A little reflection shows this must be the case, and that a regulator can only be designed in the sense that the system will operate it itself. Systems can in fact be designed as machines for self-regulation. Homeostasis is of course the example of this in biology and physiology; and thanks to Ross Ashby it is further exemplified in his kind of topological mathematics and ultrastable machines.

With these explanations, let me now try to finish my formal verbal statement. Control is the attribute of a system which tends to sustain the system's structure, to reinforce its cohesion. Control is the dynamics of the structure. To exert control in a new direction in a given system is to discover the language in which new structure may be discerned. To exert control in a recognized direction, on the other hand, is to facilitate the speaking of the language of a recognized structure. This is what communication means: the talkativeness of a structure within itself, the

ease of association inside its relatedness. So to manage the system you must be able to talk to it, to talk inside it, as a competent conversationalist.

A system, and now I am speaking emphatically of any system, can be made responsive if it is talked to in the right language, the language of its own structure which makes it the system it is. By communication inside the system, it comes to be conditioned; this is to say that the probabilities which quantify the internal relatedness of the structure begin to change. Circumstances can be created in which the system becomes conditioned to suit the behaviour we require of it; circumstances in which, with perfect propriety, the system can be said to be learning and discriminating. As management of this kind goes on paths of facilitation inside the structure become established, certain languages become more fluent than others, and certain translations from one language to another become more readily available to the system itself. In this way the system comes to exhibit memory.

At this point we have arrived knee deep in the language of physiology, and I expect a lot of complaints about this. May I at once attempt to dispose of the problem about the free use of the languages of one science in the languages of another, a practice which is always so very unpopular. When, in cybernetics, we use machine language to talk about animals; when we use biological language to discuss machines; it is then that the trouble starts. We are breaking down inhibitions of language, we are breaking through habits of thought; we are accused of a facile rhetoric, and a mere glibness of tongue. What is the point of this polyglot language, and what is the validity of the comparison it implicitly makes?

Let us arbitrarily distinguish three levels of comparison. Firstly, there is the metaphor. Perhaps when we speak of memory and learning in machines, this is the level of comparison we intend. Now the validity of a metaphor is poetic; it can offer science no more help than its verbal facility. Secondly, there is the comparison of identity itself. Perhaps when we speak of a complicated logical gate made of electronic components as a neuron we mean that it is literally the same thing as a neuron in a nervous network. But the validity of the comparison of identity between things which are not one and the same is a mystical validity. Mysticism is not generally regarded as useful to science; it may offer insight, but the vision is cloudy. Thirdly, there is analogy. This comparison has a logical validity. If we choose to say that a machine has a neurosis, perhaps we mean that its behaviour is like the behaviour of a human neurotic. Such a comparison is useful in science although it is edged about with problems

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of the delineation of its application. Any argument which we do not particularly like can be said to be "pushing the analogy too far". Nonetheless, analogical thinking is extremely useful in operational research; and I think that many operational research men who object to cybernetic discussion do so because they imagine the thinking is analogical—by analogy with operational research!

Now I am extremely anxious to assert that the physiological language of cybernetics is neither metaphorical nor analogical. Despite the mystical connotations of the assertion, I contend that the comparisons of cybernetics are identities. If a mystic were to say that two things which are manifestly not the same are the same, we would say that he had perceived, or dimly understood, some kind of ultimate identity. I am claiming that in cybernetics we are trying to pinpoint that perception scientifically. And the identity we can pinpoint is the identity of structure—as I have already defined that word. Let us take an example.

We talk about learning in the animal and the machine. It is customary to take the example of a rat running a maze for the demonstration of animal learning: the maze has two outcomes and eventually the rat learns to go to the cheese rather than the electric shock. You have perhaps heard the story of the maze-running rat who said to his colleague: "I have that chap in the white coat conditioned. Every time I run to the end of the maze he gives me a piece of cheese". But let us take this maze-learning more seriously, and consider that the maze is so constructed that there is (on a basis of random running) an equal probability of getting a piece of cheese or an electric shock. Therefore the system begins in a state of even probability over these two outcomes. And as the experiment proceeds the rat begins to approach the cheese more often than the shock until after some considerable experience he almost invariably reaches the cheese. We call this process learning; and I refuse to be impressed by the kind of special significance which some people wish to give this word. They speak of learning as if it were a prerogative of animate things; in so doing they are being dragooned by a mere word. Our rat-maze system begins in a state of even probability over the two outcomes; ultimately it approaches a state of unit probability on the cheese outcome. I choose to label this behaviour "learning". Now if we do this experiment repeatedly, we find that the initial and final outcome of the experiment, designated 0·5/0·5 and 1·0/0, are connected through time by a curve which is reproducible—and what is more a curve to which statistical significance can be given.

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Now there is the formal language and there is the abstraction of the structure of the system of learning. It is expressed in mathematical terms as a stochastic process. Once that structure is understood, there is no reason why a machine should not be built, which is a probabilistic machine, to do the same thing. And indeed, as you no doubt know, this has been done. The result is an artefact of a learning rat, which will produce experimental learning (by a Monte Carlo procedure) and provide behavioral statistics which are indistinguishable to the animal physiologist from the real results. There is an example, an elementary one, of what a cybernetician would call a learning machine. The rat running a maze is a learning machine. The mathematical specification of this stochastic process is a learning machine. The mechanical artefact is a learning machine. This illustrates quite well the idea of the structural identity of all three things. There is neither metaphor nor analogy here: there is identity in a formal sense.

So that is the kind of process that cybernetics studies, that is how it analyses what is going on, that is the heuristic mechanism of cybernetics. The result of such studies is, inevitably, that we seek to build machines: and cyberneticians have built machines that learn. They have built machines which carry out many other functions normally attributed to animate systems by a similar process of reasoning. Let me now pick up three examples of actual machines which I choose to illustrate the three main features of cybernetic systems which I began by discussing. The feature of structural probabilism is exemplified in a small way by the learning machine already described. A learning machine is essentially a conditional probability mechanism. Secondly, there is the idea of self-regulation; this may be exemplified by Ashby's homeostat,[2] where the same process of finding the structural element that is common to all self-regulating systems is pursued. In this case, the discovery is of that formal structure which leads a system to fall into an equilibrial state when perturbed by unspecified causes. And thirdly, the idea of extreme complexity is very well illustrated by one of Gordon Pask's brilliant machines[3] which formally identifies the structural elements of a process of growth.

This is what I take cybernetics to be about, and this is what cyberneticians do. But please note that because of the definitions which I have already given about systems and their structure, and the account I have given of the formal languages by which they may be described, any collection of entities which is cohesive can be described as a machine. Thus whether cybernetics deals with animals, or hardware, or any other

kind of system, is irrelevant. It is always possible to journey into the problem of describing the behaviour of a complex, probabilistic, self-regulating system in this way. And therefore there is no necessity to characterize cybernetics by the building of machines in the metal. They may equally exist in the flesh. Above all, they may arise in economic and social relationships. If the description of a social system can be mapped into the formal structure of a learning machine, then that society is a machine for learning. There is no more to be said.

2. What is Operational Research?

Having probed the nature of cybernetics, I would now like to probe the nature of operational research. I am really rather tired of the situation where, whenever anyone speaks on operational research, we have to make all these coy jokes about not really knowing what it is. What is our present approach to this question of the nature of operational research? I will tell you as I see it.

We say that this is a new and a young scientific endeavour, and that it would be very unwise to be dogmatic about it. Operational research is a sort of high-level activity; it involves studying things scientifically. Now this is a very estimable line to take. Consider how *mature* it is: one is not making any wild claims about what can be done, and what cannot be done. Consider how *safe* it is: the best of all worlds is ours to command. If someone says: "What about this operational research job, it just did not work", we may reply: "Oh, *that* was not really operational research". They may say: "Look at this wonderful job"; and we reply: "Yes—yes—*that* is operational research". Consider further how *comforting* this approach is: we sit in an esoteric coterie, hugging ourselves, and keeping warm together, which is a very superior sort of thing to do. Yes: our attitude is estimable: mature, safe, and comforting.

What price do we pay for this charming situation? We pay the price of utter confusion. There is today total confusion about the status, aims, and abilities of our work. Management has no idea of what we are trying to do, and is getting tired of trying to find out.

Recently I agreed to give a lecture on linear programming, and I was introduced by a senior and well-known industrialist who said: "Mr. Beer will give a lecture on linear programming, or as it is sometimes called, operational research". Management does not yet understand what operational research is about. Do our junior operational research people

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themselves? Most of the people at this meeting are fairly senior in operational research. I ask them what sort of future we can offer our staffs, what sort of professional status they will have in the years to come, in the brave new world? We, the Operational Research Society, have not managed to agree what operational research is, and have refused an authoritative definition to enquirers. Perhaps for this reason the Society is unable to give our staffs the professional distinction of the letters M.O.R.S. (or K.I.M.B.A.L. for that matter) to their names. And what about the national level? We have totally failed, as far as I can see, and despite some very distinguished senior men in our organization, in making anything like the right impact on the national problems to which operational research might offer a solution.

Now this is a very high price to pay for our present outlook, and I am boldly going to suggest that this outlook is simply cowardice masquerading as wisdom. For of course we know what operational research is. We have been doing it for years.

Let me quote. "Operational research may be regarded as a branch of philosophy, as an attitude of mind towards the relation of man and environment; and as a body of methods for the solution of problems which arise in that relationship". You will recognize that; it was our President speaking to the Operational Research Society.* Now I hope Professor Kendall will not think it an impertinence in me to say that I consider this is one of the best descriptions I have ever seen. It gives us an excellent start. Please notice that he did not say anything about queue theory, or search theory, or the inventory model. He gave us a description of a new man-oriented science. This is indeed what we are trying to do.

I would like to urge this point of view. Operational research is not *a* science, for it is not *about* anything; it *is* science.

Now the Newtonian universe has given place to several more sophisticated universes in succession. They are universes in which man has projected a greater maturity of his own intellect. In physics, the billiard-ball atoms have gone; in physiology, the reflex arcs have gone; in economics, the Robinson Crusoe economy has gone. We have replaced these elementary descriptions by those more advanced. In turn, operational research for me has replaced the "objectivity of science". It is the modern embodiment of "the scientific method". That phrase is one we use in the Society's

* Presidential Address on 23 October, 1958.

constitution. But what is *the* scientific method? I gather that nobody ever seems able to define that either. For me, again, it can be nothing more or less than induction, which is part of logic.

This induction has three main principles which I should like briefly to review. Perhaps we can learn from the way these three principles are being recast, what is really happening in the attack of applied science. First, there is the old principle of the uniformity of nature—associated with John Stuart Mill. This principle insisted that nature is a connected whole, within which changes occur; it led to the notion of "physical laws". By today, the same notion has led us in our work to what I have always called the operational research of "organic wholes". You all have your names for this, but the idea is the same. It is not the job of operational research to investigate the fiddling details of industrial processes, nor the pin-pricking worries of management. The operational research job is to get the problem in perspective in its proper environment; in other words, we do not suboptimize. And the second lead that the principle of the uniformity of nature gave us was the notion that what is true this time will be true next time if the conditions are the same. That notion originally led to the whole of the nineteenth century idea of the laboratory experiments, controlled experiments. And to what has that outlook matured? It has matured straight into operational research, where we have rejected the laboratory investigation with its belief that variables can be altered one at a time. We have said: real life is not like this, we have statistical techniques and the design of experiments to help us in studying more natural patterns of variation. And so this thread unfolds, until we find ourselves with the modern idea of active applied science—being precisely operational research.

Causation, the next principle of induction, is the most important exemplification of the principle of uniformity. Not only is there relatedness in the universe, there is causal relatedness. That lead, through the nineteenth century again, gave us "objectivity in science". It gave us nature as an inexorable machine, working out its answers deterministically—if only we could discover how. But the large clockwork toy has changed. The raw notion of cause has given place to the principle of indeterminacy. It was Sir Owen Wansbrough Jones whom I first heard in a broadcast talk comparing the problems of operational research with the problems of a physicist looking at the discoveries of Heisenberg. That was a useful idea, which I myself pursued in another paper.[4] We find it summed up again by Professor Kendall in his Presidential Address, when

he said: "There is a broad movement against the nineteenth century attitude towards objectivity in science". We have learnt that the observer is part of the situation. In work study there is the spotlight effect. In the introduction of new ideas there is the Hawthorne effect. These are examples of a point made earlier tonight about the observer affecting the definition of the system he seeks to describe.

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Finally, the third principle of induction: limited variety. This is the idea that properties go in sets; the idea that variety does not proliferate, as it could do in theory, by permutation and combination, but that attributes hang together in families. And for the Victorians that provided the argument from analogy. Already this evening we have had to dig ourselves out of the muddle which this testament has bequeathed. And already, I have tried to show how the concept of analogy in logic has grown and developed into the idea of structural identity. Today we have a new version of limited variety. We say: "Here is a stream of ingots coming from a melting shop, and the stream breaks up into six streams feeding six soaking pits". This model could equally well be a warehouse from which flows a stream of goods to six retail shops. Is this an analogy? No; these two things are identical in a formal sense. They are identified in a structure given by the statistics of queue theory.

In short: the universe has gone over from the motionless universe of Zeno, where nothing ever happens, to a Heracletian flux in which things are happening all the time. And the methods of science have gone from the stationary, one-variable-at-a-time kind of treatment to the treatment of operations conceived operationally—and what could that be but operational research? We no longer imagine ourselves to be using the three inductive principles to discover "what a thing is really like"; that is a pious hope belonging to the old conception. We use instead new versions of these three principles to say what makes an assembly of things tick. These new versions are essentially the operational research techniques; their object is to discover the strategy of the assembly. Thus I contend that operational research emerges as a subset of scientific methods appropriate to the analysis of *activity*, just as chemical methods of analysis from titration to spectroscopy are a subset appropriate to chemistry. And these sets of methods do not form a science—nor does operational research. And so I am saying let us do away with the idea of "*the* science of operational research"; and let us also do away with the idea "but we do not know what it is". There is really a wonderful ambivalence about these two statements anyway, is there not?

Let me now make these comments about the nature of operational research more specific by reference to a particular operational research job. I shall do this very briefly, because this is not intended as a specimen case history, but is put forward as revealing a number of important features of our work. This project identifies some of the things I take operational research to do, and which I would like to see incorporated in a definition.

The job arose in this way. What should the maximum demand for electricity be in a steelworks? Maximum demand is the greatest load of electricity you expect to call from the grid at any one time; you fix this level with your supplier in the summer, and say that you will not go beyond this figure at peak loading during the winter. The higher this figure is fixed the more safe you are, but the more expensive the contract is because the supplier has to cover himself against this load as a potential demand. The lower the figure is fixed, the less expensive the contract; but the greater risk is then run of exceeding it. Once the works comes up against its limit, it must either shed its load and thereby ruin its productivity, or must exceed its maximum demand and pay a severe fine for doing so.

We were told what the maximum demand figure operated the previous winter was; that it had in fact been exceeded more than once; and what the contractual terms and the excess levy had been. The question naturally was, what should the limit be set at now, having regard to the extra cost involved in diminishing the risk and the extra risk involved in diminishing the cost? This, you will agree, was very rightly conceived as an operational research job.

You will also agree that the main difficulty in operational research is the definition of the problem. The one we were given which I have outlined seems clear enough. But is it the real problem, or is it perhaps no more than the symptom of the real problem? We argued in this way. Electricity is a source of energy; but a steelworks has many sources of energy: coke-oven gas, blast furnace gas, steam, and oil, as well as electricity. And all these sources interact; because gas or oil can be burnt to make steam, and steam used to make electricity. Therefore, just as you can buy electricity from the national grid, you can sell electricity to the national grid (though for less money than you pay for buying it). Still further, gas can be stored in a gas holder; but if you have a contract to supply the local township with gas, the store must take that commitment into account. In fact, this is an exceedingly complex interacting system with many constraints.

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It is now clear that to consider the original problem of maximum demand in a vacuum would be to suboptimize the larger problem. Effective operational research would study the whole energy system, and this is what we did. A large scale mathematical model of the system as a whole was constructed, and from a study of this a strategy for operating the whole energy system was evolved. This strategy had then to be expressed in terms which could be understood on the shop floor. There, one cannot discuss a strategy in set-theoretic terms; one must say: "If the reading on this meter has reached this point, then providing that reading on that meter has reached that point, pull this handle". The strategy, translated into these operational terms, was accepted by the management, explained to the operatives, and installed by the operational research team.

What was the answer to the original question? From the study just described, it was possible to evaluate the optimum level for maximum demand. Far from proposing to raise the previous level, and we had originally thought that we should have to do just that, it proved possible to *halve* it. The extra power to justify this had been obtained from degrees of freedom imported to the problem from the rest of the energy system. This recommendation was accepted, the strategy adopted, and the whole job was a success.

Several critical points about an operational research study emerge from this example. First, operational research involves an environment. In the case quoted, the issue is to decide what *is* the problem and what the environment. The answer was that the other energy systems had to be incorporated in the study of the first. It is always possible to bite off an arbitrary amount of a problems environment, and to include it in the problem: the question is, how much? The first great characteristic of the operational research man is the ability to see this fact, to delineate his problem on that basis, to persuade the management that he is right, to find the method which will handle this organic structure; all this comes before the ability to solve the problem.

The important fact is that problems are simply not respecters of the official organization. A problem is first observed as a kind of outcrop; it appears in one cell of the business. There is a tendency to assume that the problem is completely contained in that cell: the cost office, a particular works department, a specific sales office. If it has ramifications elsewhere, then surely it is possible to make representations and to expect a little collaboration from "the other side". This is not the approach of operational research, which is essentially a problem-oriented activity. The

real nature of the problem must be uncovered, and this requires a very general authority to investigate the state of affairs without being contained in the organizational locality where the outcrop first appeared. If operational research men cannot be trusted with this generalized authority, then either they are not fit people for the work or the management does not understand what sort of work it is.

The specimen study concerned a dynamic interacting state of affairs which must certainly be described as both complex and probabilistic. How did operational research seize hold of the problem? Its method was to construct a model; and my next point is to ask: how ubiquitous is the idea of a model in operational research? I have often tentatively proposed that no operational research job has ever been done without the use of a model, and since this has never provoked a counter-example I will now risk the statement that this is indeed a characteristic of the work. Should anyone seek to deny this, may I warn him in advance of my defence. He says of a given job: there was no model. I reply that a model of the situation must have existed in his brain before he could do the thinking required to solve the problem. A model is no more than a description into which the real situation can be mapped. If the mapping can be done inside the skull, so much the better. But if the situation is too complicated, then the brain cannot hold the structure and its owner must go on to paper with an elaborate scheme. But the idea is the same, I contend.

The final point to be brought out is the obvious one that mixed disciplines were playing their parts in the study quoted. This point must be made but need not be laboured; for no one has ever denied that the interdisciplinary attack is characteristic of operational research.

If this is the kind of approach which we would agree constitutes operational research, there are other kinds of activity which I feel we ought steadfastly to eliminate from that classification. In the first place, to plot a set of empirical results from the works, and to note that they form a normal distribution, and that therefore a specific probability can be allocated to the likelihood that values greater than a certain parameter of the variate will occur, is not in itself operational research. It is trivial. It may be perfectly good statistics to make this observation, but it is not operational research. To ensure that I am not misunderstood to say that industrial statistics is trivial, let me take the example further. Just look at some of the models for inventory control which have been published. They are not trivial statistics, they are very advanced. But should you

come to *use* such models in the works, you find that you are speared and impaled on the assumptions implicit in the model. "Since this delay is likely to be exponentially distributed" has got into the script in small print; it seems innocuous enough. But should we plot the actual distribution as derived from an empirical study—why, it looks like the sunset silhouette of a file of camels walking down the Valley of the Kings. This is not to deny that academic model-building is useful; of course it is. But the *operational research* has hardly begun at the point when the proud author goes into print with his piece of mathematics.

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The second class of cognate activity which I should like to segregate from operational research is what I classify as erudite fun-and-gamesmanship. Consider this example of model-building for the analysis of decision procedures.[5]

N is a sequence of decision rules applied to the decision situation d. We made this specific by applying a particular starred sequence to the decisions typified as d_i; and where u is the pay-off the outcome may be written:

$$u(N^*(d_i))$$

But other decision sequences are possible; let us designate another set N^1. The first sequence is preferable to the second if their difference is positive when this has been evaluated for all i. And so we get:

$$\sum_{i=1}^{n} \{u(N^*(d_i)) - u(N^1(d_i))\} > 0$$

Which is called "The axiom of long-run success".

So far, so good. Say rather: so far and no farther. There is nothing much else we can do with this, but at least we have got a fine name. But I forget: there is another step after all. Put $\lim_{n \to \infty}$ in front of the expression, where (I quote) "the limit is assumed to exist". If I am going to take an infinite number of decisions I most certainly *expect* long-run success.

Let us have science, and mathematics; let us think deeply; let us be abstruse in our arguments where necessary. But let us relate operational research to operations and the possibility of improving them.

On the strength of this analysis, may I now attempt the definition of operational research for which I have been pleading. This one consists of two sentences including eight clauses:

(1) Operational research is the attack of modern science
(2) on problems of likelihood (accepting mischance)
(3) which arise in the management and control

(4) of men and machines, materials and money
(5) in their natural environment.
(6) Its special technique is to invent a strategy of control
(7) by measuring, comparing and predicting probable behaviour
(8) through a scientific model of a situation.

3. The Connexion

The Shared Aspects. We have reviewed the nature of the science of cybernetics, and the nature of the modern scientific method that we call operational research. I have made no attempt as we went along to relate the two, but now is the time to show how much these topics have in common.

There is a shared general intention. Cybernetics studies control as its object; operational research is normally concerned with control problems. That is implied by clause 3 of my definition. Elsewhere[6] I have tried very hard to show that operational research jobs always involve control, and that therefore they must always involve a cybernetic model. Secondly, there is the shared nature of the problem; complex and probabilistic. The only difference between the two topics in this dimension is that cybernetics stands for the problem itself, and operational research stands for the approach made to it.

Thirdly, there is the shared level of sophistication. In cybernetics, this appears as the mapping of an inanimate system into the physiological model which makes it viable. In operational research the outlook is manifested in a refusal to isolate the system from its environment, together with the important idea that the observer himself is inextricably involved in the system.

Fourthly, there is the reciprocity of method between the two subjects of which I will say more in a moment. And fifthly, there is the shared organization: which is simply the interdisciplinary attack.

So there are all these points of contact, these shared characteristics. Therefore we may next ask: is cybernetics itself the same thing as operational research? The answer is: no, it cannot be. Would any operational research man really contend that his job in life is to build thinking machines? Does operational research really have the pure structure of control as the object of study? No again: it may involve the idea of control, but the object of study is (to put the point as precisely as possible) the improvement of an operation. Nevertheless, the relationships are there, and we must consider in what way they actually work.

Operational Research (*cum Cybernetics*). In the first place, there is the notion of characteristically operational research work which has a basically cybernetic model. In our Department, there are now some seventy people. More than fifty of these are scientists and technicians, working on twenty-seven different projects at the moment. No less than eighteen of these projects are in the area of production control (as we call it). This means the field of planning, progressing, programming, stockholding, the flow of material, and so on. All eighteen projects could be described as cybernetics, if we choose to define the object as the study of control itself. But if we concentrate on the mode of attack and the general intention to improve the state of affairs, no one here would dispute that we are doing operational research.

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In being specific, I think there is no need to repeat to this audience an account of the servomechanistic model used years ago in our early work on production control in United Steel. There was an operational research attack, based on a definitely cybernetic model. Consider further, however, two more recent jobs in different works on the progressing of orders. A logical model of priority has been built which includes organic feedbacks from the plant and from the customer, so that the logic changes continuously as environmental events unfold. That is a cybernetic model. The job of quantifying the model, of transforming the logic into an arithmetic of priority numbers which can be handled as if they were natural numbers, and of installing this system to procure better deliveries, is a piece of thoroughgoing operational research. In the second works, the same problem has the different characteristic that it concerns jobbing engineering; a fact which results in a difficulty about commensurable facts. It is a typical operational research problem to find coherent measures for incommensurables, and to achieve the outcome of controlled production and accurate deliveries. But the model for this work is again entirely cybernetic. We are concerned with orders which *grow* from raw materials to completion—and growth is essentially an organic process in an organic world. It is also typified by a familiar sigmoid curve. This curve is our model, against which the growth of capital built into engineering products can be contrasted. Managerial decisions are based on fiducial control limits built on to the sigmoid curve. Finally, think of the concept, discussed elsewhere,[7] that the diversified management structure of a large industrial combine constitutes a learning machine for investing capital in stocks. There is no time to expand this point here; but I seek to implant the notion of a cybernetic machine having no hardware,

no paperwork system, no automatic data processing; a machine whose operations can be studied solely by operational research.

The basis of these ideas about operational research in production control on a cybernetic model may be found in a paper entitled "The Cybernetic Analogue of Planning and Programming",[8] which I wrote three years ago. I am very proud of this paper. The review in the *Operational Research Quarterly* was admirably succinct. It was: "Curious rather than important". Now curiosity is the essence of enquiring science; importance is the essence of conforming orthodoxy.

Cybernetics (*cum Operational Research*). Secondly, we come to the thought that in "doing cybernetics" the only methods available to the scientist are precisely those of operational research. I will give just one example of this, although the story of cybernetics is full of such illustrations. This one derives from a paper by Ross Ashby.[9]

Habituation in animals is the phenomenon whereby the response to any regularly repeated stimulus decreases. By analogy with neg-entropy it could be called neg-learning. Making a study of habituation is assuredly a job for the cybernetician: it is not an apparently operational research job. But Ashby's attack on it has many features of the operational research attack.

To begin with, he built a model of habituation, in terms of the special kind of topology he has derived from the French school known as Bourbaki. In studying this model, he came to the very surprising conclusion that habituation is a property of a wide class of general systems, and not uniquely a property of living ones as most people would suppose. Startling conclusions are typical of well-conceived operational research, and so is the result that the problem has been completely misapprehended. Moreover, Ashby finds that the habituating system is not independent of the observer: again a typical operational research result. The observer is muddled up with the system and cannot be detached. According to Ashby, to say "I tested this system for habituation, and I eventually found it to show a diminished response" is to verge on the tautologous.

As if this cybernetic study had not already revealed the operational research nature of its approach its author next proceeds to check his results by nothing other than a Monte Carlo simulation. He takes as his system a set of random numbers from Kendall and Babbington Smith, and by applying the rules of his model proceeds to show these numbers undergoing transformations in which habituation can be detected. This fittingly completes a beautiful example of the relationship I have been trying to demonstrate.

Conclusion. Now I must conclude, and I do so by answering the initial question: "What has cybernetics to do with operational research?" The answer will be given in two parts. First, as to subject matter, both topics are concerned in the main with control. This contention may be expressed in a Venn diagram (Figure 1) as follows:

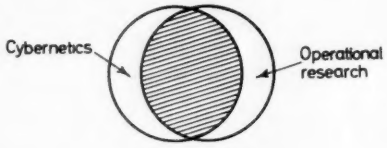

FIGURE 1.

The overlapping area is where the subject matter is concerned with control. Secondly, as to methods, I cannot conceive an operational research technique which could not be applied in the field of cybernetics, and I think that most cybernetic studies must invoke operational research techniques. This conclusion is portrayed in another Venn diagram (Figure 2):

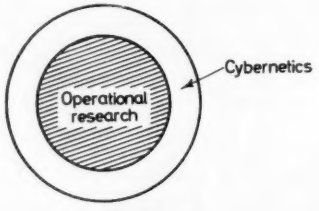

FIGURE 2.

The unshaded portion of the diagram which belongs to cybernetics alone includes such methods as the dissection of the brain and the construction of electronic machinery.

It must be made clear that these over-simplified conclusions and the diagrams are offered simply as a résumé of these arguments. They are not meant to be logically demonstrable conclusions, as are the conclusions to be drawn from Venn diagrams in logic itself. I accept that they are arbitrary; by which is meant that a means of representing the existence of a dinosaurus or the relationship of church and state could doubtless be found and added to them. They are neither unique nor logically necessary: they are there to say what I mean.

Three years ago, I did try to educe the relationship between cybernetics and operational research in a formal way. This was done through the

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concept of causation which both subjects also share, and the method used was analysis by symbolic logic. I would like to quote from that part of the paper[10] which was written in ordinary English:

> "Operational research comprises a body of methods which cohere to provide a powerful tool of investigation. Cybernetics is a corpus of knowledge which might reasonably claim the status of a science. My contention is that the two are methodologically complementary; that the first is the natural technique in research of the second, and the second the natural embodiment in science of the first."
>
> "By definition, each is concerned to treat a complex and interconnected system or process as an organic whole. By methodology, each is concerned with models and analogies from every source. By science, neither is departmental. By philosophy, each attests to the indivisible unity of knowledge."

Three years and a good deal of research in both these fields later, I am more convinced than ever that cybernetics is indeed the science of which operational research is the method. Thus, as long as this profession treats "cybernetics" as a rather dirty word, and as long as the universities in the main ignore a new science which has the audacity not to fit into the hallowed academic structure; so long will operational research practitioners, their beards growing greyer and their eyes more wild, have to pretend that they do not really know what their subject is all about.

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The Economics of the Hiring of Private Wires in Great Britain—A Simulation Study*

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The economics of hiring private telephone wires is considered in terms of the required grade of service, and the cost of the alternative public service. A maximum waiting time limitation is imposed and a simulation to determine the relationship between traffic offered to the private wire system and that accepted, is described.

1. *Introduction*

THE public telephone system in Great Britain caters for occasional telephone traffic at a cost which is proportional to the duration of the traffic. Alternatively, if the traffic between two points is sufficiently heavy it may be worthwhile to hire a private wire for the exclusive use of this traffic; the cost of hiring a private wire over a given distance is at a fixed rate per annum.

It follows that, if the hiring of a private wire is to be economically justified, the traffic it carries annually must exceed a certain break-even value—which depends only on the costs above. In most business applications, where a 5-day week is in operation, there are approximately 250 working days per year. Table 1 shows the daily break-even traffic for private wires of various lengths, based on this 250-day year.

TABLE 1

ECONOMIC BREAK-EVEN TRAFFIC ON PRIVATE WIRES

Distance (miles)	Public Service (cost/min)	Break-even Traffic (min per mile per day)
35–50	7d.	1·65
50–75	9d.	1·28
75–125	1s.	0·96
over 125	1s. 2d.	0·83

Cost of private wire is £12 per mile per year.

* Presented at the International Teletraffic Congress at The Hague, 7–11 July, 1958.
† Now with Shell Mex & B.P. Ltd., London.
‡ Now with I.C.I. (Heavy Organic Chemicals) Ltd., Billingham, Co. Durham.

The break-even traffics are of the order of 50–300 min/day. The typical working day, say 9.00 a.m. to 5.00 p.m. is only 8 hr long but allowing for the relatively slack periods at the beginning and end of morning and afternoon, it is doubtful whether the effective period of use of a private wire would exceed 5–6 hr.

If the required break-even traffic is to be carried in only 5–6 hr the traffic intensity will be sufficiently high to cause considerable delays, giving rise to a relatively poor grade of service. It is not economic to improve the grade of service by hiring additional private wires. If the grade of service is to be satisfactory it is necessary to place some limit on the waiting time of calls, and this is done by specifying that, after waiting a certain time for service on the private wire system, calls shall be transferred to the public service.

One consequence of the waiting time limitation is that the private wire system will not carry all the telephone traffic generated during the day. The problem we consider is the relationship between the traffic accepted by the private wire system and the traffic offered to it, under various conditions of traffic generation and waiting time limitation.

The problem is not one of statistical equilibrium, and this, together with the waiting time limitation, excludes an analytical solution in the general case. In this paper we describe an experimental solution by simulating the operation of a private wire system on a digital computer. The only reason for the use of a digital computer was that the amount of computation required was beyond the reasonable scope of hand computation.

2. *Simulation*

2.1 *Synthetic Traffic*

2.1.1 *Parameters*

The first stage of the simulation was the production of synthetic telephone traffic with predetermined statistical properties. The statistical properties of the traffic were as follows.

(i) *Call Arrival Pattern.* The two sources, one at each end of the private wire system, each generated calls at random, with prescribed rates. Consequently the combined generation of calls was at random at a rate which was the sum of the rates from the individual sources, and the intervals between the arrivals of successive calls had a negative exponential distribution with a mean depending on the rate of call generation.

The call generation rates were not constant throughout the day; there were instead, well defined:

"slack periods": 9.00 a.m.–9.30 a.m., 12 noon–2.00 p.m., 4.30 p.m.–5.00 p.m.

"busy periods": 9.30 a.m.–12 noon, 2.00 p.m.–4.30 p.m.

These periods were the same for both sources, and were based on a survey of actual telephone traffic.

(ii) *Call Holding Times.* The call holding time distribution, also based on a survey of actual telephone traffic, was the same for both sources and is tabulated in Appendix 2. The average holding time was 8 min.

Provision was made for changing some of the parameters controlling the statistical properties of the traffic, viz.

$\bar{x}$ = the average traffic generated in min/day

γ = the ratio $\dfrac{\text{traffic generation intensity in busy periods}}{\text{traffic generation intensity in slack periods}}$

r = the ratio of the traffic generation rates from the two sources.

No provision was made for altering the distribution of call holding times or the average call holding time. Provided the shape of the distribution was not altered, any change in the average would affect only $\bar{x}$ which was already more easily controlled by the rate of call generation.

2.1.2 *Production*

The computer programme was arranged to generate a new call every time the arrival of a call was registered. To do this three pseudo-random numbers were generated (see Appendix 1). These numbers were integers in the closed interval [00, 99].

The first was used to sample from a basic inter-arrival interval (negative exponential) distribution stored in the computer. The distribution of the intervals and the correspondence between intervals and pseudo-random integers is shown in the table in Appendix 2. The mean of the stored basic inter-arrival interval distribution was 1 min and the sample interval was multiplied by the actual mean interval (determined by the values of $\bar{x}$ and γ and whether the system was in a busy or slack period) to give the actual interval.

The second pseudo-random integer was used to allocate the source of the call according to the ratio r. The procedure of generating only one set of calls (at the rate of the two sources combined) and then allocating the calls to one of the two sources by a suitable random process can be shown, for calls generated at random, to be equivalent to separate generation for each source at the rate appropriate to each source. It has, moreover, the practical advantages of requiring only one call generating mechanism and of automatically resolving any difficulties arising from the simultaneous

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arrival of calls from the two sources.

The other pseudo-random number was used to sample from the holding time distribution (see Appendix 2) exactly as the first was used to sample the basic inter-arrival distribution.

2.2 *The Handling of the Synthetic Traffic*

The computation required to handle the synthetic traffic involves only the straightforward arithmetic of determining, from their holding times, when calls currently in service will have completed service, and of determining the waiting times of any calls queueing and transferring (to public service) those which exceed the waiting time limit. An outline of the organization of this arithmetical procedure is given in Appendix 1.

In selecting calls for service and allocating them to private wires, however, certain rules of priority between the sources had to be observed.

(i) *Priority Procedure.* (a) When a call was to be selected for service and there were queues of calls from both sources the rules were as follows. With an odd number of private wires, one wire was designated the "common" wire and on the common wire priority was given to calls from the source which did not generate the last call served on it. With an even number of wires, half gave absolute priority to one source and half to the other. When questions of priority had been resolved selection was on the basis, first come, first served.

(b) When there were calls queueing from only one source these calls were served, in order of preference, on wires giving absolute priority to their source, the common wire (if any), and finally the remaining wires.

2.2.1 *Parameters*

Provision was made for changing one parameter controlling the handling of the synthetic traffic, viz.

Q—the maximum waiting time (in minutes) of a call before its transfer to the public service.

2.3 *Data Derived from the Simulation*

The principal data from the simulation were, the traffic generated and the traffic carried by the private wire system, which was simply the total duration of the calls served by the system.

The traffic generated was more difficult to define. It was clearly the traffic carried by the private wire system plus the traffic transferred to the public service. The difficulty was in the measurement of the traffic transferred. At first sight it might appear that the sum of the durations of the

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calls transferred was required; in fact this quantity played no part in the operation of the system and to use it only introduced into the figure for traffic generated an unnecessary sampling fluctuation. The best figure for traffic transferred was the number of calls transferred multiplied by the average holding time and this has been used.

In addition to the principal data, certain other figures were produced. They were the number of calls served on the private wire system and the number transferred to the public service, the total waiting time of calls served on the private wires (calls transferred are known to have waited Q min each) and the number of delayed calls carried by the private wire system. All of these are quantities of interest to the user of a telephone system.

Furthermore, certain of the figures are, or can be, used to derive values of $\bar{x}$, the average holding time and the number of calls generated per day for the synthetic traffic sample and, by comparison with the average values introduced into the simulation, they give a check on the behaviour of part of the sampling procedure. In fact, all the above data were obtained separately for each of the combinations of the two sources and busy/slack periods, thus permitting checks on γ and r also.

3. *Choice of Parameter Values and Traffic Samples*

The complete simulation experiment comprises a large number of trials, each trial consisting of one day's simulated operation with fixed values of the parameters. We consider here the choice of parameter values and the number of days of simulated operation required for each combination of parameter values.

The parameter values have been chosen with the intention of covering the region of operating conditions which is likely to be of interest, in such a way that results for any situation arising in practice can readily be found by interpolation. In studying the field to be covered, approximate analytical solutions for $Q = 0$ and $Q = \infty$ are helpful. The approximate solution for $Q = \infty$ is calculated on the assumption that the private wire system will carry all the traffic generated up to the limit of its capacity. The approximate solution for $Q = 0$ is calculated on the assumption of statistical equilibrium, random arrival and negative exponential holding time, in which case it is well known that the state probabilities have a truncated Poisson distribution. These solutions for $Q = 0$ and $Q = \infty$ are plotted in Figures 1, 2 and 3 (Appendix 3) together with the experimental results.

These results for the limiting values of Q, together with the range of break-even traffic shown in Table 1, narrow down the experimental field very appreciably. Even so, a large number of parameter combinations is required to cover it and at the time of writing this paper the full experiment has not been completed. Work has so far been limited to the development of the computer programme and a relatively small number (150) of simulated day's operation (trials) in order to study the most suitable parameter values and the variability of individual day's traffic samples.

There are five parameters, the four introduced above and additionally the number of private wires in the system. The region studied has been that around an average traffic generated of 640 min/day, where the results for $Q = 0$ and $Q = \infty$ indicate that two private wires may be more economical than one. This choice of parameter values was fairly obvious; the choice for γ and Q was less so. (It was considered that in the initial trials there was no need to change r at all and it was kept throughout at 1 : 1.)

For exploratory purposes the experimental values of γ were chosen to be (1, 4·2, ∞) and of Q (6, 16); (the experimental levels of Q being supplemented by the analytical solutions for 0 and ∞). It is not, in fact, the absolute values of Q which are important but their ratio to the average holding time, which determines the general time scale of the system. The effectiveness of this choice of levels is considered later in relation to the results of the initial trials.

In all 30 parameter combinations, listed in Table 2, were used in the initial trials.

TABLE 2

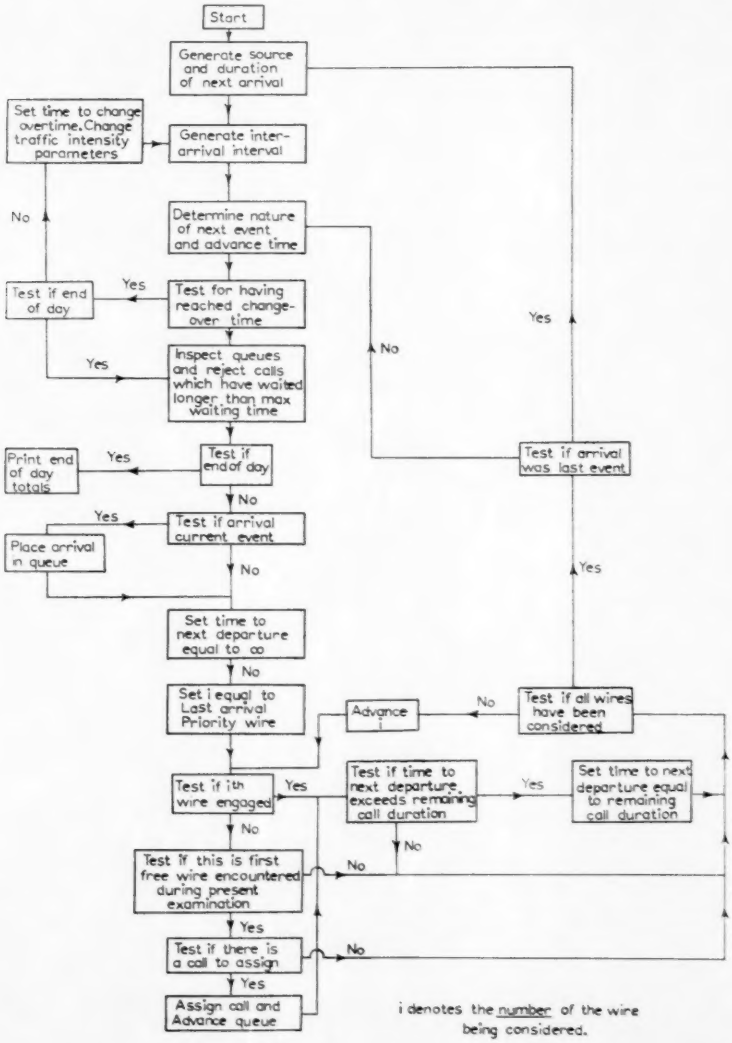
PARAMETER COMBINATIONS IN THE INITIAL TRIALS

Parameter	Symbol	Levels	
Average traffic generated	$\bar{x}$	480 min	800 min
No. of private wires	n	1, 2	1, 2, 3
Busy/slack intensity ratio	γ	1, 4·2, ∞	1, 4·2, ∞
Max. waiting time	Q	6, 16 min	6, 16 min
Number of combinations		12	18

Each of the 30 parameter combinations was the subject of 5 separate trials, giving the total of 150 trials. The choice of traffic samples for these trials required consideration. One possibility was to generate a different traffic sample for each of the 150 trials. The disadvantage of this scheme

was that the effects of different parameter values would be inseparable from the effects of different traffic samples.

It was advantageous to be able to test different combinations of parameter values on the same traffic samples. The use of pseudo-random


Start
Generate source and duration of next arrival
Set time to change overtime. Change traffic intensity parameters
Generate inter-arrival interval
Determine nature of next event and advance time
No
Test if end of day
Yes
Test for having reached change-over time
Yes
Inspect queues and reject calls which have waited longer than max waiting time
No
Yes
Test if arrival was last event
Print end of day totals
Yes
Test if end of day
No
Yes
Test if arrival current event
Place arrival in queue
No
Yes
Set time to next departure equal to ∞
No
Set i equal to Last arrival Priority wire
Advance i
No
Test if all wires have been considered
Test if ith wire engaged
Yes
Test if time to next departure exceeds remaining call duration
Yes
Set time to next departure equal to remaining call duration
No
No
Test if this is first free wire encountered during present examination
No
Yes
Test if there is a call to assign
No
Yes
Assign call and Advance queue
i denotes the number of the wire being considered.

numbers to generate the samples made this possible for certain parameters. Changes in the number of private wires (n) and maximum waiting time (Q)

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do not affect the traffic sample at all and the same sample could be reproduced for any values of these parameters. Changes in r and γ have only minor effects on the traffic generated, although they affect the pattern of generation, and even if it is not possible to reproduce exactly the same traffic samples when the values of these variables are changed, provided the same pseudo-random numbers are used the samples will be much more alike than independent samples. There is no possibility of reproducing the same traffic samples for different values of average traffic generated.

The full 150 trials have therefore been arranged in what is effectively a split-plot design. At each level of average traffic there are 5 different traffic samples and all parameter combinations are tested on each traffic sample (as far as this is possible for γ). Although the split-plot design is well suited to the analysis of results by an analysis of variance, in the present case further complications are introduced by the variation of the actual traffic generated about the average value. In this connexion it has been found that, even for a fixed traffic sample, because of changes in the identity of calls carried and transferred and the method of assessment of traffic offered there are considerable differences between treatment in the actual traffic offered. This can be clearly seen from the results for the individual trials shown in Appendix 3. If it were wished to make a statistical assessment of the precision of the results it would be necessary to correct for the variations in traffic offered by means of an analysis of covariance. However, having regard to the errors arising in the practical measurement of average traffic levels, we are satisfied that the results are of sufficient precision for the purpose of assessing the economics of private wire hire.

4. *Results*

The numerical results are tabulated in Appendix 3. It will be noticed that there is a clearly marked trend for traffic offered to increase with γ. This we ascribe to a greater loss of traffic at the beginning and end of the day occurring with the lower values of γ since these give more traffic at these times.

The main results are given in Table A.3.3 and are shown, together with the analytical solutions for $Q = 0$ and $Q = \infty$ in Figures 1, 2 and 3. The general impression is that they are much as would be expected from the limiting analytical solutions and that the intermediate values chosen for Q and γ do in fact divide the region of interest fairly evenly.

The results also suggest that the busy/slack traffic intensity ratio (γ), which is in practice unlikely to be outside the range (2, 6), will have little

effect on the economics of a private wire system. As far as the maximum waiting time is concerned it is clear that little is to be gained by extending the limit to much more than twice the average call duration (i.e., to more than 16 min in the present case).

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ACKNOWLEDGEMENTS

We would like to thank Mr. S. Stonehewer of I.C.I. Central Instruments Laboratories for the computer programme and the description in Appendix 1, and I.C.I. Ltd. for permission to publish this paper.

Appendix I

Programme Description

The arrival time (in minutes past 9.00 a.m.) divided by 1,000 and the duration (in minutes) of all calls are stored as fractional and integral parts respectively of numbers in certain "arrival locations".

A location is also reserved for each wire and in this location is stored the remaining call duration of the call at that time occupying the wire.

During each examination of the wires "time to next departure" is set equal to the least positive remaining call duration. Before each examination "time to next arrival" is compared with "time to next departure" and time is advanced to the event occurring first.

The mid-square method for generating random numbers was employed. An n digit binary number is squared and from the resulting $2n$ digits n are extracted. These n digits are then used for the next operation and repetition does not occur before approximately 10^{15} operations have been completed.*

Each day's simulation takes approximately 24 sec made up as follows:

General computation	– 6 sec
Random number operations	– 6 sec
Output of results	– 12 sec.

Appendix II

Special Checks on Programme Operations

It has been mentioned in Section 2.3 that some of the sample data produced by the simulation programme can be used to carry out checks on the sampling procedure.

* D. F. Votaw, Jr. and J. A. Rafferty, "High Speed Sampling", *Mathematical Tables and Aids to Computation* **5**, 4 (1951).

It was felt, however, that more detailed confirmation of the operation of the programme was also required and certain more detailed checks were carried out during the development of the programme. These comprised a check on the handling of the synthetic traffic and checks on the sample distributions of holding time and inter-arrival time.

TABLE A.2.1

(A) INPUT BASIC INTER-ARRIVAL INTERVAL DISTRIBUTION

Basic Interval (min)	Frequency	Corresponding Random Integers
0·05	10	00–09
0·15	9	10–18
0·25	8	19–26
0·35	7	27–33
0·45	6	34–39
0·55	6	40–45
0·65	5	46–50
0·75	5	51–55
0·85	4	56–59
0·95	4	60–63
1·10	6	64–69
1·30	5	70–74
1·50	5	75–79
1·70	4	80–83
1·90	3	84–86
2·10	3	87–89
2·40	2	90–91
2·70	2	92–93
3·10	2	94–95
3·50	1	96
4·10	1	97
4·70	1	98
5·40	1	99
Total	100	

Mean = 1·00 min ; Variance = 1·04

A special programme was used for test purposes. The parameter values were chosen to give operating conditions which would test all parts of the programme, while the programme itself was designed to print out a large amount of data over and above that normally required. These data comprised full details of the time of arrival, source, time of service and waiting time of each call, together with tabulated histograms of the sample distributions of holding times and inter-arrival interval.

The handling of the synthetic traffic was checked by a straightforward hand computation of the traffic sample (170 calls) generated during one simulated day. Full agreement as regards the identity of the calls carried by the private wire system and the calls transferred, and the times of service and waiting times was required from this check.

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TABLE A.2.1

(B) INPUT HOLDING TIME DISTRIBUTION

Duration (min)	Frequency	Corresponding Random Integers
2	3	00–02
3	12	03–14
4	13	15–27
5	13	28–40
6	12	41–52
7	8	53–60
8	6	61–66
9	5	67–71
10	5	72–76
11	4	77–80
12	3	81–83
14	6	84–89
17	4	90–93
20	3	94–96
24	2	97–98
30	1	99
Total	100	

Mean = 8·00 min; Standard Deviation = 5·35 min.

Tables A.2.2 (A) and (B) show the observed and expected distributions, the calculated χ^2 and the corresponding confidence probabilities.

The sampling procedure for the generation of the synthetic traffic was checked both by comparison of the sample parameter values with their average values which were part of the programme input, and by comparison of the sample distributions with their input distributions. These checks were carried out on the results of 4 days' simulation.

The tests of parameter values were based on the assumption of a normal distribution of sample parameter values (this was justified by the sample sizes). No significant deviations from the average were found.

The results of the χ^2 tests on sample distributions are given in more detail. Tables A.2.1 (A) and (B) show the input distributions of basic

inter-arrival interval and holding time together with the random integers which, in the simulation, corresponded to each interval.

TABLE A.2.2

(A) χ^2 TESTS ON INTER-ARRIVAL INTERVAL SAMPLES

Day	1		2		3		4		Total	
Interval \ Frequency	Obs.	Exp.	Obs.	Exp.	Obs.	Exp.	Obs.	Exp.	Obs.	Exp.
0·05/0·15	33	31·7	23	27·6	52	38·0	30	29·9	138	127·3
0·25/0·35	25	25·1	19	21·8	26	30·0	30	23·6	100	100·5
0·45/0·55	18	20·0	20	17·4	33	24·0	17	18·8	88	80·4
0·65/0·75	21	16·7	14	14·5	18	20·0	13	15·7	66	67·0
0·85/0·95	11	13·4	16	11·6	14	16·0	13	12·6	54	53·6
1·10	13	10·0	9	8·7	14	12·0	10	9·4	46	40·2
1·30	6	8·4	4	7·3	5	10·0	7	7·9	22	33·5
1·50	5	8·4	8	7·3	7	10·0	6	7·9	26	33·5
1·70	8	6·7	6	5·8	9	8·0	5	6·3	28	26·8
1·90	5	5·0	3	4·4	3	6·0	8	4·7	19	20·1
2·10	8	5·0	5	4·4	5	6·0	4	4·7	22	20·1
2·40	5	6·6	10	5·8	9	8·0	5	6·3	15	13·4
2·70									14	13·4
3·10	6	5·0	3	4·4	3	6·0	5	4·7	12	13·4
3·50									5	6·7
4·10									6	6·7
4·70	3	5·1	5	4·4	2	6·0	5	4·7	2	6·7
5·40									7	6·7
Total	167		145		200		158		670	
χ^2	8·26		8·94		18·01		6·02		12·57	
Degrees of Freedom	13		13		13		13		17	
P	0·83		0·78		0·16		0·94		0·77	

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TABLE A.2.2

(B) χ^2 TESTS ON HOLDING TIME SAMPLES

Day	1		2		3		4		Total	
Holding time \ Frequency	Obs.	Exp.	Obs.	Exp.	Obs.	Exp.	Obs.	Exp.	Obs.	Exp.
2	14	18·6	23	18·8	15	20·5	29	19·4	17	15·5
3									64	61·8
4	10	16·1	16	16·3	19	17·8	16	16·8	61	67·0
5	18	16·1	23	16·3	26	17·8	21	16·8	88	67·0
6	15	14·9	16	15·0	16	16·4	8	15·5	55	61·8
7	15	9·9	8	10·0	13	11·0	14	10·3	50	41·2
8	4	7·4	3	7·5	8	8·2	6	7·7	21	30·9
9	7	6·2	7	6·3	4	6·9	6	6·5	24	25·8
10	12	6·2	4	6·3	6	6·9	6	6·5	28	25·8
11	9	8·7	9	8·8	13	9·6	3	9·1	22	20·6
12									12	15·5
14	9	7·4	10	7·5	5	8·2	6	7·7	30	30·9
17	2	5·0	3	5·0	5	5·5	5	5·2	15	20·6
20									15	15·5
24	9	7·4	3	7·5	7	8·2	9	7·8	6	10·3
30									7	5·2
Total	124		125		137		129		515	
χ^2	13·27		12·04		9·73		15·86		18·34	
Degrees of Freedom	11		11		11		11		15	
P	0·27		0·36		0·55		0·15		0·25	

The above significance tests show that the sampling procedure is producing satisfactory samples of the input distributions.

Appendix 3

Results of the Simulation

TABLE A.3.1

TRAFFIC OFFERED (min/day)

$\bar{x}$	γ	Q	n	Sample 1	Sample 2	Sample 3	Sample 4	Sample 5	Mean
	1	6	1	405	376	524	363	456	425
	1	6	2	364	426	538	341	457	425
	1	16	1	397	417	518	350	449	423
	1	16	2	362	427	525	336	452	420
	4·2	6	1	371	430	507	399	497	441
480	4·2	6	2	368	450	539	373	474	441
	4·2	16	1	379	484	509	387	504	453
	4·2	16	2	362	467	517	365	465	435
	∞	6	1	402	493	560	437	467	472
	∞	6	2	397	520	564	416	492	478
	∞	16	1	391	520	539	444	507	480
	∞	16	2	396	538	546	414	497	475
				Sample 6	Sample 7	Sample 8	Sample 9	Sample 10	Mean
	1	6	1	673	776	795	864	752	772
	1	6	2	687	773	764	910	810	789
	1	6	3	681	801	793	907	805	797
	1	16	1	685	757	753	867	790	770
	1	16	2	708	773	757	884	791	783
	1	16	3	687	795	802	901	797	796
	4·2	6	1	721	763	781	934	838	807
	4·2	6	2	702	767	817	936	847	814
800	4·2	6	3	701	795	856	927	823	820
	4·2	16	1	714	810	769	940	834	813
	4·2	16	2	701	779	846	937	811	815
	4·2	16	3	686	794	864	956	801	820
	∞	6	1	698	799	835	930	832	819
	∞	6	2	694	817	853	893	821	816
	∞	6	3	696	805	880	941	822	829
	∞	16	1	691	842	822	916	840	822
	∞	16	2	695	776	843	932	849	819
	∞	16	3	710	808	857	973	827	835

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TABLE A.3.2

TRAFFIC CARRIED (min/day)

$\bar{x}$	γ	Q	n	Sample 1	Sample 2	Sample 3	Sample 4	Sample 5	Mean
	1	6	1	285	272	340	267	320	297
	1	6	2	356	402	514	333	449	411
	1	16	1	331	313	390	294	393	344
	1	16	2	362	419	525	336	452	419
	4·2	6	1	243	278	283	279	321	281
	4·2	6	2	352	402	475	357	442	406
480	4·2	16	1	299	300	341	315	360	323
	4·2	16	2	362	435	501	365	449	422
	∞	6	1	234	245	272	253	275	256
	∞	6	2	341	408	460	392	428	406
	∞	16	1	263	288	299	308	315	295
	∞	16	2	380	474	522	414	463	451
				Sample 6	Sample 7	Sample 8	Sample 9	Sample 10	Mean
	1	6	1	369	376	387	416	408	391
	1	6	2	567	621	612	710	666	635
	1	6	3	633	745	745	843	765	746
	1	16	1	445	445	417	475	454	447
	1	16	2	620	693	677	812	727	706
	1	16	3	687	787	778	885	797	787
	4·2	6	1	345	363	365	382	374	366
	4·2	6	2	566	583	593	640	623	601
800	4·2	6	3	653	707	760	815	727	732
	4·2	16	1	410	402	425	404	418	412
	4·2	16	2	629	651	662	689	683	663
	4·2	16	3	678	754	816	916	793	791
	∞	6	1	306	311	283	306	296	300
	∞	6	2	518	545	525	573	557	544
	∞	6	3	616	701	696	781	686	696
	∞	16	1	339	338	318	332	320	329
	∞	16	2	575	608	587	612	609	598
	∞	16	3	662	744	761	861	771	760

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TABLE A.3.3

TRAFFIC CARRIED BY EACH ADDITIONAL PRIVATE WIRE

$\bar{x}$	γ	Q	Traffic Carried			Traffic Offered		
			1st PW	2nd PW	3rd PW	1st PW	2nd PW	3rd PW
480	1	6	297	114	—	425	425	—
	1	16	344	75	—	423	420	—
	4·2	6	281	125	—	441	441	—
	4·2	16	323	99	—	453	435	—
	∞	6	256	150	—	472	478	—
	∞	16	295	156	—	480	475	—
800	1	6	391	244	111	772	789	797
	1	16	447	259	81	770	783	796
	4·2	6	366	235	131	807	814	820
	4·2	16	412	251	128	813	815	820
	∞	6	300	244	152	819	816	829
	∞	16	329	269	162	822	819	835

Note that the figures for traffic offered in the above table are those corresponding to the larger of the two numbers of private wires involved, e.g. the figures for the second PW are those for $n = 2$ in Table A.3.1.

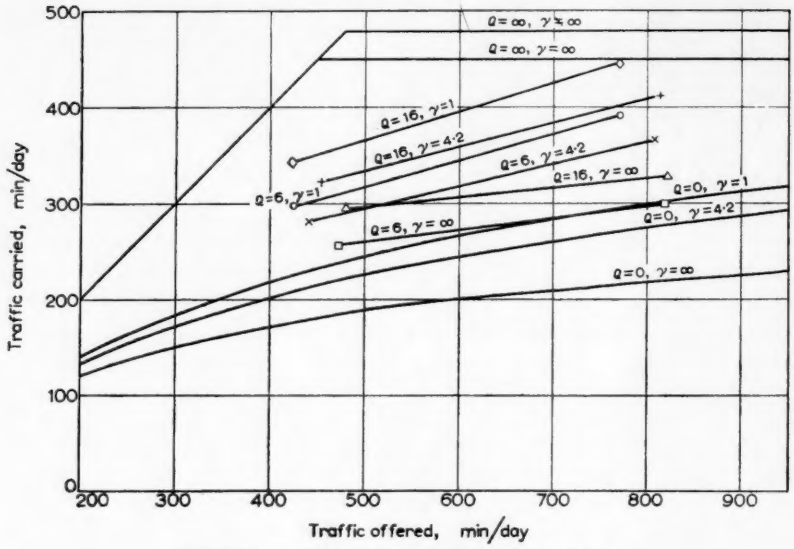

FIGURE 1. *Traffic carried on first private wire.**

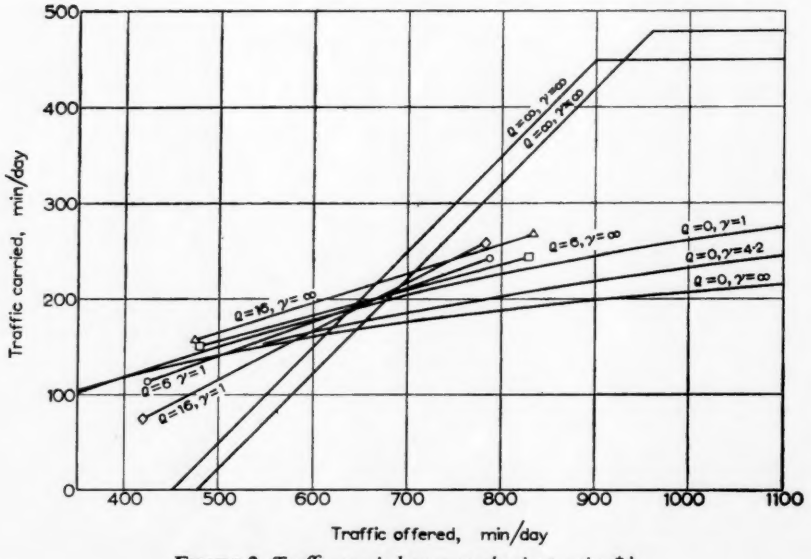

FIGURE 2. *Traffic carried on second private wire.**†

* The straight lines joining the experimental points are only to indicate the linking of the points and are not intended to be used for interpolation.

† In Figure 2 the results for $\gamma = 4{\cdot}2$ have been omitted in the interests of clarity.

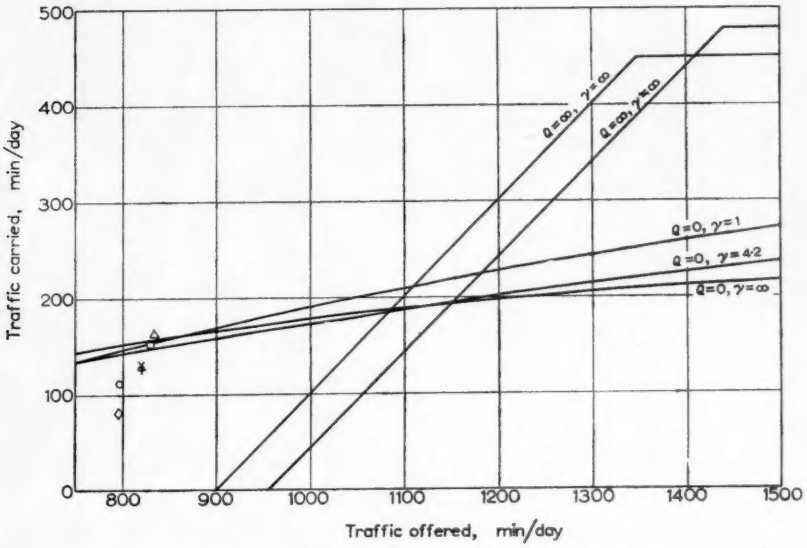

FIGURE 3. *Traffic carried on third private wire.*

KEY TO SYMBOLS USED IN FIGURES 1–3

- ○ $Q = 6$, $\gamma = 1$
- ◇ $Q = 16$, $\gamma = 1$
- × $Q = 6$, $\gamma = 4{\cdot}2$
- \+ $Q = 16$, $\gamma = 4{\cdot}2$
- □ $Q = 6$, $\gamma = \infty$
- △ $Q = 16$, $\gamma = \infty$

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TABLE A.3.4

OTHER RESULTS

$\bar{x}$	γ	Q	n	Percentage of Calls transferred to P.S.	Percentage of Calls Delayed	Av. Waiting Time—Delayed Calls only
	1	6	1	28·88	61·01	4·34
	1	6	2	3·21	22·50	2·97
	1	16	1	17·75	73·19	8·88
	1	16	2	0·36	23·57	4·39
	4·2	6	1	35·09	71·93	4·40
480	4·2	6	2	7·72	33·68	3·32
	4·2	16	1	22·82	64·51	10·51
	4·2	16	2	2·81	36·49	5·41
	∞	6	1	45·30	82·88	4·81
	∞	6	2	15·10	49·33	3·70
	∞	16	1	38·93	89·93	11·90
	∞	16	2	5·03	56·38	6·22
	1	6	1	49·90	82·18	4·89
	1	6	2	20·00	51·87	3·88
	1	6	3	6·65	26·20	3·48
	1	16	1	42·98	93·19	11·30
	1	16	2	10·06	61·64	7·66
	1	16	3	1·25	29·52	5·20
	4·2	6	1	56·21	85·13	4·93
	4·2	6	2	27·03	63·21	4·23
800	4·2	6	3	11·18	40·65	3·39
	4·2	16	1	51·22	92·45	12·83
	4·2	16	2	19·35	73·72	9·58
	4·2	16	3	3·66	49·39	6·36
	∞	6	1	65·32	91·12	5·18
	∞	6	2	34·27	75·81	4·51
	∞	6	3	16·73	54·84	3·79
	∞	16	1	62·10	97·98	13·76
	∞	16	2	27·82	87·50	10·50
	∞	16	3	9·48	65·32	7·52

Some Approximate Statistical Tests

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This paper enumerates and discusses a number of quick, short-cut statistical tests that are available for two common problems. The first problem is testing the difference between the means of samples drawn from two populations whilst the second is testing a sequence of observations to see whether there is a trend present.

In a recent issue of this journal Duckworth and Wyatt[1] have given a number of quick and approximate methods that may be useful to operational research workers. Some further techniques have been given in a paper by Watkins[2]. These authors pointed out that short-cut methods are often useful for making a preliminary appraisal of data or for providing a check against gross errors in any full-scale analysis that may have been carried out.

This paper extends and discusses some of the tests given in the above papers for the comparison of two samples or for the presence of a trend in a series of observations.

The Two Sample Problem

We assume that we have two random samples drawn from normal populations each having the same standard deviation. Sample A has n_1 individuals, sample B has n_2 individuals. A test suggested by Duckworth and Wyatt when n_1 and n_2 are both equal to n is to pair the two samples off in a random order and to count the number of times the individual from A is greater than the corresponding individual in B. The number of such positive differences is distributed as the binomial $(\frac{1}{2}+\frac{1}{2})^n$ when there is no difference in the population means from which the two samples come.

This test suffers from three disadvantages:

(a) It loses a certain amount of information since there is just one artificial pairing,

(b) it cannot be used for unequal sample sizes,

(c) with small samples it is difficult to establish significance because the exact values of the observations are not used but merely the sign of the difference.

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An alternative test proposed by Tukey and described by Duckworth and Wyatt meets (a) and (b) to a large extent but is still unsuitable for small samples. The test consists in arranging the two samples in order of magnitude and counting the total number of observations in the non-overlapping portions of the two samples. If this total exceeds 7, or 8 if the sample sizes are very unequal, the difference is judged to be significant.

Mood's Test

A further alternative test, based on Mood[3], is to find the joint median of the two samples combined and then to count the number of individuals in sample A that fall above the median, a, say. On the null hypothesis that both A and B are samples from the same population a is a hypergeometric variable. If $n' = \frac{1}{2}(n_1+n_2)$ and we assume n' is integral the probability distribution of a is

$$p(a|n_1, n_2) = \frac{{}^{n'}C_a \, {}^{n'}C_{n_1-a}}{{}_{n_1+n_2}C_{n_1}}$$

$$= \frac{n_1!\, n_2!\, (n'!)^2}{(n_1+n_2)!\,(n'-a)!\,a!\,(n_1-a)!\,(n'-n_1+a)!}$$

If n_1 and n_2 are both above about 10 it is usually sufficiently accurate to assume that a is normally distributed with mean $\frac{1}{2}n_1$ and standard deviation

$$\tfrac{1}{2}\sqrt{[(n_1 n_2)/(n_1+n_2-1)]}$$

Table 1 gives the significant values for a. The bottom left hand part of the table gives the 5% level whilst the top right hand part of the table

TABLE 1

5% AND 1% SIGNIFICANT VALUES OF a

	SAMPLE B (n_2)						
SAMPLE A (n_1)	8	10	12	14	16	18	20
8	7/**7**	**8**	**9**	**11**	**12**	**13**	**14**
10	8	8/**9**	**10**	**11**	**12**	**13**	**14**
12	9	9	9/**10**	**11**	**12**	**13**	**14**
14	10	10	10	10/**11**	**12**	**13**	**14**
16	11	11	11	11	11/**12**	**13**	**14**
18	12	12	12	12	12	12/**13**	**15**
20	13	13	13	13	13	14	14/**15**

To use this table make $n_1 \geqslant n_2$ and look up roman type for 5% limit whilst for 1% limit make $n_2 \geqslant n_1$ and look up **bold type**.

gives the 1% level. Sample A is always taken as the larger sample if the two samples are of unequal size. It should be noted that the use of this table errs on the conservative side. That is to say the significance level actually used never exceeds the stated 5% or 1% and is quite frequently lower. This occurs since a is a discrete variable and the sum of the tail probabilities of the distribution of a cannot take every possible value between 0 and 1 but only certain values.

Example 1. This is taken from the paper by Watkins. The median of the 28 values is found, quite straightforwardly, to be 9·9. In sample *A* 10 values, marked with an asterisk, are above the median and, from Table 1, this is significant at 5% but not at 1%. When Duckworth and Wyatt's test was applied, however, an unfortunate pairing of the observations gave a non-significant result.

Sample A	*Sample B*
9·5	9·4
9·1	9·6
9·0	9·2
9·5	8·0
12·2*	8·9
10·0*	8·5
10·7*	9·7
10·4*	11·1
11·5*	9·4
11·2*	11·4
11·6*	10·9
10·2*	11·3
11·8*	9·8
12·0*	9·3

t-*Test Based on Range*

Most of the difficulties can be overcome by using the standard t-test but replacing the standard deviation by the range. The criterion used is

$$u = \frac{|\bar{x}_1 - \bar{x}_2|}{w_1 + w_2}$$

where $\bar{x}_1$, $\bar{x}_2$ are the two sample means and w_1, w_2 are the ranges in the two samples. Lord[4] proposed this test and gave significance values for $2u$

when the two sample sizes are equal. In a recent journal Moore[5] has given significance levels for u for all values of n_1 and n_2 between 2 and 20. This test, although it needs a small table of significance levels, is extremely quick and simple to apply. Moore shows, in the paper quoted, that the loss of effectiveness over the conventional form of t-test using standard deviation is negligible for small sample sizes.

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The three tests discussed as being suitable for unequal sizes of samples will now be applied to an example.

Example 2. For a high-voltage grid, cables of great but equal tensile strengths are needed. Each cable is composed of wires which are originally manufactured in one length. In order to measure the tensile strength of two cables a number of wires are taken from each of the two cables and their tensile strengths measured in kilograms.

Wire *A* 340, 330, 325, 328, 338, 332, 335, 340, 336, 341.

Wire *B* 347, 341, 345, 340, 350, 346, 339, 338.

Test (*i*) Using Tukey's proposed test we count the number of observations in wire *B* that are above the highest observation in wire *A* (341) and add to this the number of observations in *A* that are below the lowest observation in *B* (338). This gives $6+3=9$ and as this is above 7 (8 or 9 for very unequal sample sizes) the difference is significant.

Test (*ii*) For Mood's test we find the median to be $\frac{1}{2}(339+340)$ or 339·5. In wire *A* there are 7 values below the median (or 3 above) and from Table 1 this is significant at 1%.

Test (*iii*) To use Lord's modified t-test based on range we calculate

$$u = \frac{|\bar{x}_1 - \bar{x}_2|}{w_1 + w_2} = \frac{|334{\cdot}5 - 343{\cdot}25|}{16+12} = 0{\cdot}312$$

From the tables given by Moore this is significant at 1%. If the usual form of t-test based on standard deviation is used we find $t = 3{\cdot}654$ with 16 degrees of freedom. From tables the two-ended probability that t, with 16 degrees of freedom, is outside $\pm 3{\cdot}654$ is 0·0024. Hence the difference is significant at 1% and the three tests above all agree with this conclusion.

Test for Trend

A very simple test, for a monotonically increasing or decreasing trend in a series of observations $x_1, x_2, x_3 \ldots\ldots x_{n-2}, x_{n-1}, x_n$ can be carried out as follows.

Observe whether x_n is greater or less than x_1. If x_n is greater than x_1 examine whether x_{n-1} is greater than x_2. If so examine whether x_{n-2} is greater than x_3 and continue this process until the opposite result occurs. If x_n is less than x_1 examine whether x_{n-1} is less than x_2 and so on until the reverse occurs. Let r be the number of pairs that have the same sign for the difference before a pair occurs with the opposite sign. The probability that r is greater than or equal to an amount r_0 is equal to 2^{-r_0} for the two-ended test, that is testing for a trend that is either increasing or decreasing. For a one-ended test the appropriate probability is 2^{-r_0-1}. Hence for a one-ended test r must be 6 or more for the trend to be significant at 5% whilst if r is 8 or more the trend is significant at 1%.

Example 3. Does the following sequence, taken from Duckworth and Wyatt's paper, exhibit a positive trend?

22, 21, 37, 25, 39, 46, 29, 10, 49, 45, 15, 7, 57, 44
23, 28, 59, 50, 40, 31, 41, 24, 47, 51, 42, 47, 55, 47

From these values we find x_{28} is greater than x_1. Also x_{27}, x_{26}, x_{25}, x_{24}, x_{23} are all greater than their corresponding earlier values but x_{22} is less than x_7. Hence r is equal to 6 and the trend is significant at 5% but not at 1%.

This conclusion differs from that obtained by Duckworth and Wyatt who used a procedure proposed by D. R. Cox. For this method the last nine values are compared with the first nine values. That is x_{20} with x_1, x_{21} with x_2 and so on up to x_{28} with x_9. For significance to be established in this case either 8 or 9 of the differences must have the same sign. In this case there are 6 pluses and 3 minuses and the trend is not significant. It is probably just an unfortunate pairing that has produced the non-significant result since the 6 plus difference values average 21·3 whilst the three minus differences average only 6·3. The bias is very definitely positive and a t-test between the two sets of figures indicates a significant difference. It should be pointed out that had Cox's test been applied to the complete set of 28 observations and x_1 compared with x_{15}, x_2 with x_{16} and so on, there are 11 positive differences and 3 negative differences. Taking the difference $11-3$, 8, we find that this just exceeds $2\sqrt{14}$ and hence the difference is significant at 5% using Duckworth and Wyatt's test.

This example would seem to be a case where there is not an overwhelming trend and whilst some tests will bring it out others will not. If there is a regular monotonic trend—which is probably not the case in the example just discussed—the powers of the various tests can be worked out. Power in this context is the probability that a given test will detect the presence

of a trend—alternatively it can be defined as the probability that it rejects the hypothesis that there is no trend present. The test due to Cox appears to be the most powerful one whereas the test proposed above has a somewhat lower power, rather in line with the power of tests proposed by Stuart and Foster[6] that were based on the records occurring in a sequence of observations.

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Trend Test Using Median

The alternative test given by Duckworth and Wyatt to detect general systematic variations in the population mean uses the median of the N observations. If we put plus for all observations above, and minus for all observations below the median the test is then based on G, the number of changes of sign that occur in the ordered sequence of N signs. It should be noted that if the (assumed unknown) population median were used to determine which sign is to be given to an observation the distribution of G would follow the binomial

$$(\tfrac{1}{2}+\tfrac{1}{2})^{N-1}$$

However as the number of plus signs and minus signs are each fixed at $\frac{1}{2}N$ the successive signs are not entirely independent of each other. Also G must be at least one and cannot be zero as the binomial distribution would imply to be possible. To calculate the exact distribution we proceed as follows.

Let $N = 2n$. The number of ways of arranging n plus signs and n minus signs to form $2t$ or $2t+1$ sets are respectively

$$f_{2t} = 2\left\{\frac{(n-1)!}{(t-1)!\,(n-t)!}\right\}^2 \quad \text{and} \quad f_{2t+1} = f_{2t} \times \frac{n-t}{t}$$

where t takes the values 1, 2, ...n. The probability, if the observations are arranged at random, of obtaining a sequence of n plus signs and n minus signs arranged in r sets is therefore

$$p_r = f_r \Big/ \sum_{\text{all } r} f_r$$

where r is taken as $2t$ or $2t+1$ above as is appropriate. Now since G will always be equal to $r-1$ we have that the probability distribution of the number of changes G is equal to

$$p_G = f_{G+1} \Big/ \sum_{\text{all } r} f_r$$

This distribution is compared in Table 2 with the binomial approximation above for the three cases of N equal to 10, 20 and 30. The distributions are all symmetrical, although the centre point for the exact and approximate distributions differ by a half. The lower portions of the distributions for N equal to 20 and 30 can, therefore, be deduced from the upper portions. It will be seen that there are fairly large variations in the individual probabilities. However in carrying out tests of significance we are more interested in sums of tail probabilities and Table 3 summarizes the situation for the three values of N used here. The table gives the 5% and 1% values of G obtained from the approximate and the exact distributions, together with, in brackets, the *exact* probabilities obtained from Table 2 of getting the stated or a lower value of G. These values are

TABLE 2

COMPARISON OF APPROXIMATE AND EXACT DISTRIBUTIONS FOR G

G	$n = 5$ $N = 10$		$n = 10$ $N = 20$		$n = 15$ $N = 30$	
	Approx.	Exact	Approx.	Exact	Approx.	Exact
0	0·00195	—	0·00000	—	0·00000	—
1	0·01758	0·00794	0·00004	0·00001	0·00000	0·00000
2	0·07031	0·03175	0·00033	0·00010	0·00000	0·00000
3	0·16406	0·12698	0·00184	0·00088	0·00001	0·00000
4	0·24609	0·19048	0·00739	0·00351	0·00004	0·00002
5	0·24609	0·28571	0·02218	0·01403	0·00022	0·00011
6	0·16406	0·19408	0·19048	0·03274	0·00088	0·00043
7	0·07031	0·12698	0·05175	0·07638	0·00291	0·00171
8	0·01758	0·03175	0·14416	0·11457	0·00799	0·00470
9	0·00195	0·00794	0·17620	0·17186	0·01865	0·01292
10	—	—	0·17620	0·17186	0·03730	0·02584
11	—	—	0·14416	0·17186	0·06444	0·05168
12	—	—	0·09611	0·11457	0·09666	0·07752
13	—	—	0·05175	0·07638	0·12641	0·11627
14	—	—	0·02218	0·03274	0·14446	0·13288
15	—	—	0·00739	0·01403	0·14446	0·15187

TABLE 3

NOMINAL SIGNIFICANCE VALUES FOR G AND THE EXACT ASSOCIATED PROBABILITIES

Significance level		5%	1%
$N = 10$	Approx.	1 (0·001)	0 (0·000)
	Exact	2 (0·040)	1 (0·008)
$N = 20$	Approx.	5 (0·019)	4 (0·005)
	Exact	5 (0·019)	4 (0·005)
$N = 30$	Approx.	9 (0·020)	7 (0·002)
	Exact	10 (0·046)	8 (0·007)

always, of course, below the nominal 0·05 or 0·01 owing to the discrete nature of G. It is clear that the approximate distribution is unnecessarily severe, that is it tends to make the probabilities associated with rejection even further below the nominal 0·05 and 0·01 than the exact distribution does. The first two moments of the two distributions are

Approximate mean	$\frac{1}{2}(N-1)$	Exact mean	$\frac{1}{2}N$
Approximate variance	$\frac{1}{4}(N-1)$	Exact variance	$\frac{N(N-2)}{4(N-1)}$

In the limiting case, when N is very large, the two variances are both equal to $\frac{1}{4}(N-1)$ but the means will always differ by $\frac{1}{2}$ so that the two distributions can never coincide exactly.

The test for runs using G (or the number of runs, $G+1$) has been discussed by Stevens[7], David[8] and Moore[9] and it has been shown that the normal distribution provides a reasonable approximation to the distribution of G for large N. The mean and variance to be used would be those just given above.

Conclusions

This paper has discussed the utilization of quick and easy statistical methods for two problems, first the comparison of two samples and secondly the testing of a sequence of observations for a trend. Although for some of the tests simple tables of significance levels are required the tests all possess the merits of being extremely simple to apply and they do not require calculating machines.

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Competitive Bidding*

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This paper considers the problem of making simultaneous bids for a number of objects against competition by an unknown number of competitors. A method of solution is given which has been successfully applied in practice.

1. The Problem

IN this paper we shall be considering the derivation of solutions to problems involving the submission of competitive bids for properties or rights or competitive tenders for services. The work is based on an actual case study. This study was carried out in a highly competitive industry. We are disguising (successfully we hope) the type of industry and the particular circumstances of the problem itself.

Suppose we have a situation where a number of people are bidding for a number of objects and where the values of the objects will not be known precisely until after they are bought. Simultaneous bids are invited on a given day for some or all of the group of objects, the bids are kept secret until the closing time, all bids are then declared and each object goes to the highest bidder for that particular object.

For example:

TABLE 1

BIDS RECEIVED FOR A SET OF OBJECTS

Person Bidding	A	B	C	D	E	Winner
Object:						
a	50	40	75	—	—	C
b	20	30	—	15	—	B
c	90	80	85	75	95	E
d	—	—	70	60	50	C
Total amount bid	160	150	230	150	145	
Total amount spent	0	30	145	0	95	

In time, as successive data on bids received at successive sales are declared, it is possible to build up a picture of the behaviour of the various bidders. The problem as posed was:

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If in a sale, bids are invited for n objects, and if my estimate of the values of these n objects is $v_1, v_2, \ldots v_n$, and if I have a total amount S to put out on bids, how should I spread this in order to get the maximum expected gain, measured by estimated value minus cost?

Let $p(b_j, _j)$ be the probability of a bid of b_j on the jth object actually winning.

Then we have to minimize the total expected gain

$$G = \sum_{j=1}^{n} p(b_{j,j})(v_j - b_j)$$

Subject to

$$\sum_{j=i} b_j = S$$

2. *Earlier Work on the Subject*

The early work on this subject was done by Friedman while at Case Institute of Technology[1] and the present authors have been fortunate in being able to develop their study from Friedman's pioneering work in the field.

In brief, Friedman derived relationships between the number of competing bidders on a given object and the (estimated value of) average bid submitted for the same object.

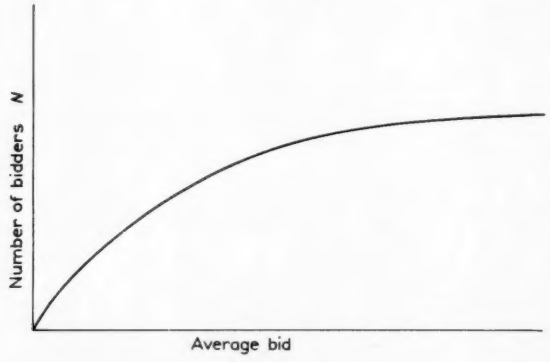

FIGURE 1.

He also derived, for a large number of sales, a "well behaved distribution" of the ratio of all the individual bids (successful and unsuccessful) for each object to the average bid for this object (Figure 2). Finally, it was possible to derive a relationship between the average bid in a sale and the estimated value of the object.

Hence it was now possible to estimate the chance of a given bid succeeding on an object with a given estimated value in the following manner:

(1) Determine estimated value (v).
(2) Use this estimated value to derive the expected average bid (a).
(3) Use a in Figure 1 to estimate the number of bidders (N).

The probability of a bid of b succeeding is then (by reference to Figure 2), p^N.

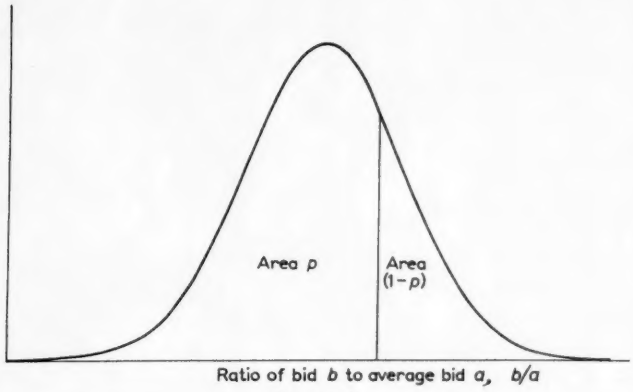

FIGURE 2.

If now we have a set of objects of estimated values $v_1, v_2, \ldots v_n$, it is possible to calculate the probability of success for any nominated bid b, on a given object of value v. If the probability of winning is P, then the expected return is $P(v-b)$. We have to maximize $\Sigma P(v-b)$ subject to Σb = constant.

For example:

TABLE 2

EXPECTED RETURN, $P(v-b)$, FOR DIFFERENT BIDS

Object:	A	B	C
0	0	0	0
5	1	4	3
10	3	8	10
15	10	15	25
20	15	25	22
25	17	35	18
30	12	40	14
35	8	35	10
40	2	25	7
45	−3	18	2
50	−10	12	−4

Total bids to equal 25.

We solve this by iteration. Choose a trial solution, say 10, 15, 10: the expected return is $3+15+10 = 28$. We note that an incremental increase

of 5 in the C bid will give a greater gain than in any of the others while an incremental decrease of 5 in the A bid will give a smaller loss than any of the others. Hence we take as our next solution 5, 15, 15; the expected return is now $1+15+25=41$. The next step in the iteration gives the final solution (0, 20, 15) with the expected return 50. In general, the number of objects for a sale in the area studied is of the order of 80. The iteration process is still feasible. If two people work together, the solution by iteration can be obtained within less than 1 hr.

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3. *Shortcut Method for Determining the Probability of Winning*

Friedman determines the probability of winning a particular object by separately estimating the probability P of beating a single competitor ("average bidder") and the number N of competitors. The estimate of N can be a source of considerable inaccuracy. (The same holds true of an estimate of the expected value of N when a probability distribution is

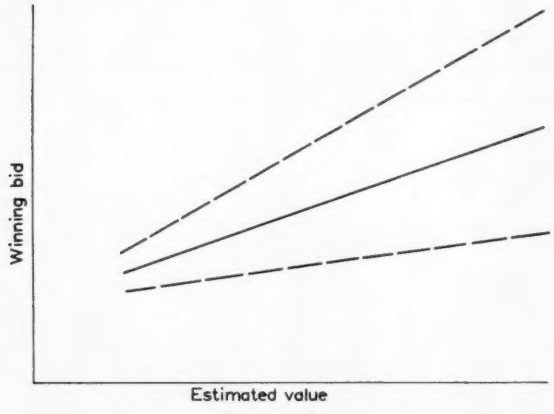

FIGURE 3.

used.) Moreover, since both estimates are ultimately based on the estimated value of the object and on relations derived from history, the authors felt that it was desirable to derive from history, a direct relationship between the estimated value of an object and the probability of winning it with a bid of a given amount. A plot of winning bids as a function of estimated value revealed a definite correlation, although with considerable variance

about the regression line (Figure 3). The variance was not constant. This result suggested an analysis of the ratio:

$$\frac{\text{Winning bid}}{\text{Estimated value}} = \frac{w}{v}$$

which is plotted as a function of the estimated value itself in Figure 4. This time no correlation resulted, and it was concluded that the ratio w/v is a random variable with a probability distribution independent of v. In some cases there may be a non-constant variance about the horizontal

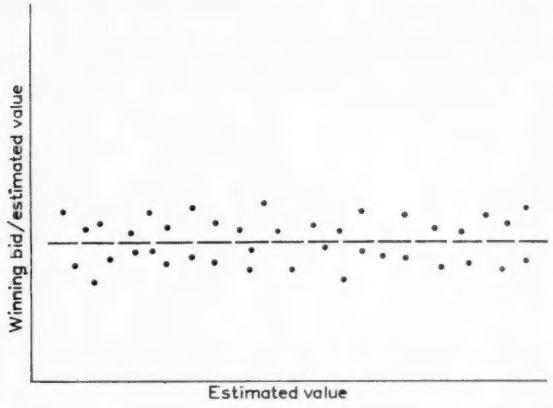

FIGURE 4.

regression line in Figure 4. Then it becomes necessary to derive separate probability distribution for various value ranges. (This case is not considered in the present paper.) The probability density turned out to be the logarithmic-normal type:

$$f\left(\frac{w}{v}\right) = \frac{1}{\sqrt{(2\pi)}\sigma} \exp\left[-\frac{\left(\log\frac{w}{v} - \mu\right)^2}{2\sigma^2}\right]$$

The parameters μ and σ could be estimated very satisfactorily from the data. The general shape of the curve remains as in Figure 2. However, the interpretation has changed; the area under the curve to the left of a given point x/v now directly represents the probability of winning an object of estimated value v with a bid of amount x. This probability is $F(x/v)$ where $F(x)$ is the cumulative function of $f(x)$. This approach ensures not only a

considerable gain in accuracy but also reduces to a fraction the amount of computational work required for setting up the tables of expected return.

4. Relative Measure of Value

In our case study we frequently encountered difficulties in obtaining estimates of money value of the objects under discussion whereas a relative measure of value in "points", a_i, could be obtained with relative ease. In order to handle these situations, the objective of bidding was slightly changed, and the problem was reformulated as follows: How should a given amount of money be allocated in the form of bids to n objects in order to maximize total expected value won, measured in "points"? Or, mathematically speaking, maximize

$$E(V) = \Sigma a_i F\frac{(b_i)}{a_i}$$

Subject to $\Sigma b_i = \text{const.}$

This formulation results in a further reduction of computational effort since the subtraction of the bid from the estimated value is eliminated.

5. Price Trends

So far, we have made the tacit assumption that history will repeat itself; actually, the parameters μ and σ show a time trend. Particularly μ, the average price paid for one unit of value, depends greatly on general economic and other factors. By a linear regression analysis of μ on these factors a forecaster of μ was found. The variability of σ was of lesser importance and is not discussed here.

6. Further Work

One of the interests of this problem from the research point of view is the possibilities of useful further study. In the first place we might consider the "power" of the Friedman method and of the method proposed in this paper. Consider the case where the same general type of bidding situation arises a large number of times. If x and y represent respectively the total profit of applying the two different methods to this large number of bidding situations, then we shall have built up to a distribution of x and y, e.g. $f(x)$ and $g(y)$. It would be of interest to solve explicitly for f and g. Alternatively, f and g could be derived by Monte Carlo methods, and on a computer this would not be difficult.

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We might also in similar fashion derive a power curve for alternative methods of solution. An alternative method which obviously suggests itself is that of dynamic programming. In comparing different methods one needs to stipulate either the maximum amount of money which can be bid, or the maximum amount of money which one should expect to pay out on successful bids. The solutions proposed both by Friedman and ourselves use the former deterministic restriction and it would be interesting to see results using the latter probabilistic restrictions.

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Queueing Processes Associated with Airline Passenger Check-in

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Passenger check-in systems at airports and town terminals are outlined and shown to be equivalent to combinations of, at most, four queueing processes of different types. The determination of the optimum system by using both theoretical and empirical methods is discussed. This is a case study, the results of which have been applied.

Introduction

THE first thing which an airline passenger must do on reporting for his flight at any airport or town terminal is to check in; that is, register his baggage, and exchange his flight coupon for a boarding pass. As at any moment, passengers may be checking in for several different flights, a number of manned service (i.e. check-in) positions must be provided.

Unless there is an excessive number of check-in positions, passengers must occasionally wait for service. From the passenger's viewpoint there is therefore a grade of service associated with check-in measured by the time he must wait to be served plus the time taken to serve him. The airline however, is not only interested in pleasing the passenger but must also operate economically. Consequently when new check-in areas are being planned or old ones modified, two questions must be answered. First, what is the minimum grade of service which should be provided? Second, which type of check-in system will provide this grade of service at least cost?

The definition of the minimum grade of service which is acceptable is a management responsibility, as commercial policy, assessment of the market for air travel and study of passengers' reaction must all be considered. The specification of the check-in system, which will provide this grade of service most economically is, however, largely an exercise in applied probability theory.

This paper describes an operational research study which was carried out to determine the best type of check-in system to adopt at certain BEA stations. The lessons of this study have already been applied, with satisfactory results, at several airports and terminals.

Preliminary Studies and Field Work

The terminology used in practice to describe the main features of check-in systems is confused, but in this paper terms are standardized as follows.

(1) *Flight open* is the earliest time before flight departure at which passengers are allowed to check in for a flight.

(2) *Flight deadline* is the published time before flight departure by which passengers must arrive (i.e. join a check-in queue) to be *entitled* to check in for a flight.

(3) *Flight complete* is the time before departure at which check-in of the last passenger to be accepted has been completed.

(4) *Flight close-out* is the latest time before departure at which check-in of a passenger for a flight may be begun. After close-out no more passengers are allowed to check in and the flight is said to have closed.

These definitions are important and must be noted.

The first stage in the investigation was to study existing check-in systems to gain an insight into practical problems of operation. It was learnt that there were two basic types of check-in systems in use. Under the first, one service position was provided for each flight for a fixed period preceding the scheduled time of departure, and all passengers for a flight had to check-in at the corresponding desk. This may be called the *completely restricted* system. Under the second system, however, any passenger could check in for his flight at any one of a number of desks up to "close-out" time. This is known as the *common check-in* system.

Under a flight check-in system, as the completely restricted system is often called, passengers arriving before flight open time cannot check in, and must disperse in the lounges, restaurants and so on, later returning to check in when their flight is open. When a common check-in system is in force, early passengers can check in immediately. Common check-in systems, however, have one serious disadvantage. If a passenger arrives to check in for his flight just before the deadline, he might find himself in a queue at a common check-in desk behind passengers for other, later flights. The clerk would not know which flight the passenger wished to join until he had reached the front of the queue, by which time the flight might possibly have closed. This is a very undesirable state of affairs which could not occur with a completely restricted system of checking in flight by flight, because the clerk would know that any passengers queueing at

his desk were for his flight. If it became clear that he could not check in all passengers in the queue before close-out, he would call upon an additional clerk—known as a "booster"—for help.

A compromise could be adopted to overcome these difficulties. Under such a system, passengers reporting for their flights up to a certain time before close-out (referred to here as *transfer time*) would check in at common desks: those arriving later than the transfer time, but before close-out, would check in at restricted desks, known in such circumstances as final check-in desks. Transfer time is so called because at that time, check-in would be transferred from common desks to final desks. This may be called a *composite* check-in system.

During these preliminary investigations a very important limiting condition was determined. The time at which check-in of the last passengers for a flight has been completed (the *flight completion time*) must not be later than 5 min after close-out time, or there would be roughly a 1 in 4 risk of a late flight departure. The cost of late departures is so great that no check-in system which led to an increase in their frequency could be considered. Furthermore, the managements of the two principal stations involved in the studies agreed, independently, that an average passenger waiting time of 0·5 min at busy times would correspond to the minimum acceptable grade of service.

It was possible at this point to define the problem more closely. The greatest element of cost in running any of these systems is labour cost, and the most economical system is that which needs fewest operating staff. The optimum check-in system therefore would be that which would require the minimum number of staff to:

(i) give an average passenger waiting time of 0·5 min during the busy hours of the day

(ii) ensure that for only 5 flights in 100 was the completion time later than 5 min after close-out time.

It was clear that under these circumstances the determination of an optimum check-in system was equivalent to the determination of the optimum transfer time for a composite check-in system.

The second stage in the investigation was to examine the labour requirements of a composite check-in system. It was found that four main types of clerical work had to be carried out:

(i) Checking in passengers at common desks. This involves weighing and labelling baggage, writing in weights in flight coupons, making out excess payment vouchers (if any), issuing boarding passes and

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transmitting flight coupons to a control room for sorting and checking.

(ii) Checking in passengers at final (restricted) desks. This involves all the operations performed at common desks plus entering baggage weights on to a load control sheet, maintaining a count of the number of passengers and of the weight of baggage so far checked in and a few other smaller jobs.

(iii) Controlling: Sorting of tickets from common desks into flights.

(iv) Controlling: Adding and listing by machine, the number of passengers and baggage checked in for each flight at common desks, and then reconciling the baggage weights.

The time expended on each type of work, per passenger, had to be obtained.

Observation showed that the distributions of service times at the two types of check-in desk were as shown in Figure 1 with the mean values:

(i) Common check-in: 1·64 min per passenger group.

(ii) Final (restricted) check-in: 2·5 min per passenger group.

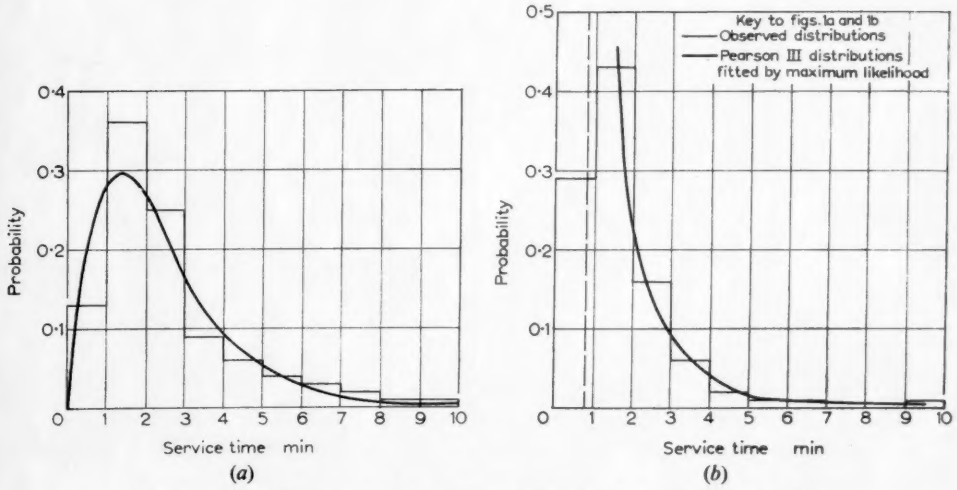

FIGURE 1. *Distribution of service times.* (*a*) *Restricted check-in (final desks).* (*b*) *Common check-in.*

A passenger group consists of a number of passengers travelling together, with common baggage (mean size 1·5 persons). The distributions are of the Pearson type III family, as shown by the continuous curves in Figure 1.

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The third and fourth types of work could not be observed directly because there was no composite system in operation which involved a sufficient amount of common check-in to require special control facilities. Consequently controlled experiments were conducted to obtain the required data.

The ticket-sorting rate was estimated in the following manner. A cardboard mock-up was made of the types of ticket-sorting rack envisaged for future use. A factorial experiment, arranged in randomized complete blocks in which the factors were traffic clerks (3), arrangements of sorting rack (4), and the blocks days (3), was carried out to determine the time taken to sort 160 flight coupons for 16 destinations. It was discovered that the best arrangement of sorting rack gave rise to a mean sorting time of 2·1 sec per ticket, with a coefficient of variation of less than 5 per cent for individual sorting times.

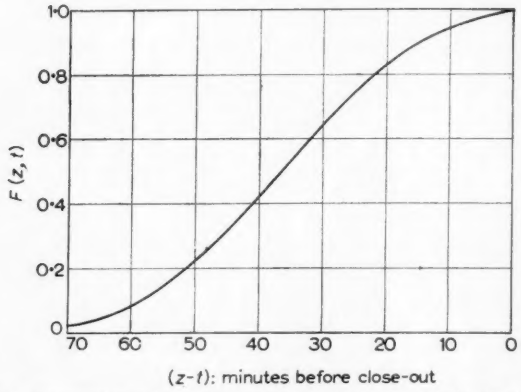

FIGURE 2. *Proportion of passengers arriving by z_0 for flight closing at t.*

The time taken to addlist n passengers and their baggage, reconcile baggage weights and transmit the totals to Load Control, was estimated in a similar manner and found to be given on average by $(0{\cdot}5n+4{\cdot}5)$ min, with ignorable variance.

Thus the service times for the 4 queueing processes which had been found to constitute a composite check-in system were determined; but to apply queueing theory, the inputs too were required. These were obtained by recording the times at which passenger groups reported for their flights. One result is illustrated by Figure 2 which shows the proportion $F(z, t)$ of passengers for a flight closing-out at time t who arrived by

time z. (All times are measured backwards from flight departure time.) Further results obtained by statistical analysis of the data were:

(i) *Common check-in*: during a busy hour when flights are departing at regular intervals, passenger groups arrive at common check-in desks in a Poisson stream at a mean rate of

$$\frac{1}{a} = \frac{F(z_0, t)}{60} \sum_{i=1}^{k} n_i \text{ passenger groups per min}$$

where z_0 is the transfer time, k the number of flights departing in the hour and n_i the number of passenger groups on the ith flight.

(ii) *Restricted check-in*: For the ith flight, the number of passenger groups who would arrive during the period (z_0, t) would have the expected value $n_i(1 - F(z_0, t))$ and be distributed binomially.

(iii) *Ticket sorting*: Tickets would arrive from common desks to the sorting position approximately in a Poisson stream at a mean rate of $(1{\cdot}5/a)$, a being defined above.

(iv) *Addlisting*: The input of batches of flight coupons to the addlisters would be regular at 1 batch every $(1/k)$ hours.

In all four queueing processes, the order of service was seen to be usually "first come, first served" although observation convinced us that at times a more complex queue discipline was in operation.

At this point, all the information needed to build up a theoretical model was to hand, and the construction of such a model was the next stage in the project.

The Theoretical Model

The model which was studied was a composite check-in system as already described, with the parameters obtained during the field studies and certain assumptions regarding workload, which were:

(i) *Throughput*: During a busy hour, one flight would be scheduled to depart every 5 min.

(ii) *Passenger loads*: For each flight 20 passenger groups (or 30 passengers) would arrive before close-out time.

The whole model could be summarized in Table 1.

For a fixed transfer time z_0, these queueing processes can be considered independently.

The statistical difficulty associated with the first process is that little appears to be known about multiserver queueing processes with Pearson III

service-time distributions. There are a number of theoretical results for some particular cases but where these are convenient for numerical applications they relate to a small number of servers ($n = 1$, 2 or 3) or to chi-square distributions with even numbers of degrees of freedom.

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TABLE 1

SUMMARY OF TRAFFIC PARAMETERS FOR MODEL

Queueing Process	Aspect	Type	Average
Processing Functions			
(i) Common desks	Input	Poisson	4 $F(z_0, t)$ passenger groups per min
	Service	Type III	1·64 min per passenger group
(ii). Final desks	Input	Binomial	$20/(z_0 - t)\{1 - F(z_0, t)\}$ passenger groups per min per flight
	Service	Type III	2·5 min per passenger group
Control Functions			
(iii) Coupon sorting	Input	Poisson	6 $F(z_0, t)$ coupons per min
	Service	Constant	0·035 min/coupon
(iv) Addlisting	Input	Regular	1 flight every 5 min
	Service	Binomial	$1{\cdot}5\ F(z_0, t) + 4{\cdot}5$

Consequently, recourse is occasionally made to Monte Carlo methods to solve particular cases of such problems; but possibly more often it is assumed that a bold approximation, using random arrivals and negative-exponential service times, will provide answers accurate enough for practical purposes.

In this study both procedures were adopted and the results compared. These comparisons proved so interesting that further results were calculated using different Pearson III service-time distributions obtained in other studies. The results of this preliminary analytical work are briefly described in the Appendix to this paper. The conclusion reached was that very satisfactory results could be obtained quickly without recourse to expensive Monte Carlo simulations by estimating the mean waiting time $\bar{w}_n$ for n servers from the formula:

$$\bar{w}_n = \tfrac{1}{2}E_n(1 + V^2)$$

where E_n is the mean waiting time for n servers if the service-time distribution were negative-exponential with the same mean, and V is the coefficient of variation of the Pearson III service-time distribution.

This formula, which is due to Pollaczek is exactly true for $n = 1$ (Kendall[1]). This conclusion was of practical value since, of course, it is comparatively easy to evaluate E_n using Erlang's formula (Fry[2]):

$$E_n = P(0)\frac{\epsilon^n b}{(n-\epsilon)^2(n-1)!},$$

b being the mean service time and where

$$\frac{1}{P(0)} = \frac{\epsilon^n}{(n-\epsilon)(n-1)!} + \sum_{r=0}^{n-1}\frac{\epsilon^r}{r!}$$

and ϵ = number of arrivals during an average service time.

Thus the number of servers at common desks required to give a mean passenger waiting time of about 0·5 min could be quickly calculated.

For the particular model under consideration, the number of service clerks required, corresponding to various transfer times was found to be as given in Table 2.

TABLE 2

NO. OF CHECK-IN CLERKS REQUIRED AT COMMON DESKS FOR DIFFERENT TRANSFER TIMES

Transfer time in minutes before close-out $(z_0 - t)$	5	10	15	20	25	30
n	8	8	8	7	6	5

At common desks the restrictions on completion times is not important unless $z_0 = t$ (i.e. there are no final desks) or alternatively the probability of a passenger having to wait longer than $(z_0 - t)$ min becomes appreciable.

There is another failing of common check-in systems to be avoided which has already been mentioned. This is the risk that a passenger who has joined a queue prior to close-out time for his flight, may not reach a check-in desk to be served until after that time. This introduces a further restriction. These two restrictions may be re-expressed in an equivalent way, introducing limits of risk as follows:

(i) The chance that any passenger who has joined a queue to check in for his flight before transfer, is not able to obtain service until after close-out must be so small that it happens on not more than 5 per cent of flights.

(ii) The chance that check-in of any passenger which has commenced before close-out, has not been completed by 5 min after close-out must be so small that it occurs on not more than 5 per cent of flights.

Consider the first restriction. The calculation of the risk of occurrences of this type would involve the waiting time distribution. But this was not known, and there seemed no possibility of obtaining it theoretically in the time available. (It still does not appear to have been derived at the present time.) Thus very precise results were unobtainable.

The Monte Carlo simulations, based on a completely common check-in system ($z_0 = t$) had however generated waiting time distributions, and these are shown, for various numbers of common check-in positions in Figure 3. It is apparent that for a composite system, when the transfer

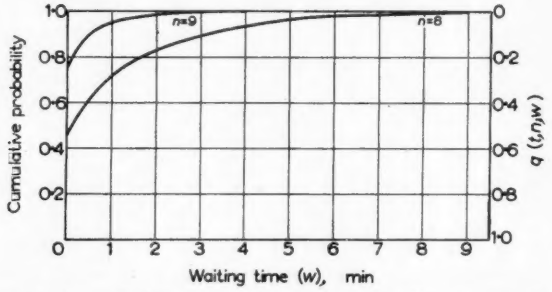

FIGURE 3. *Cumulative probability distributions of waiting times for completely common systems* ($z_0 = t$).

time is z_0 and the input to common desks is less, the probability $q(z_0, n; w)$ of a waiting time of w or longer given that there are n servers must satisfy:

$$q(z_0, n; w) \leqslant q(t, n; w)$$

where t is as usual, close-out time.

The probability that a passenger arrives during the interval $(u, u = \mathrm{d}u)$ is say $P(u)\mathrm{d}u$; so the unconditional probability that he arrives before z_0 and does not reach a service position until after t, is:

$$Q(z_0, n) = \int_{z_0}^{\infty} p(u)\, q(z_0, n;\, u - t)\, \mathrm{d}u$$

$$\leqslant \int_{z_0}^{\infty} p(u)\, q(t, n;\, u - t)\, \mathrm{d}u = Q^*(z_0, n)$$

The probability that at least one passenger group on a flight will arrive before transfer time yet not reach a check-in position until after that time is not less than

$$\{1-[1-Q^*(z_0, n)]^{20}\}$$

It is easily shown using the waiting time distributions of Figure 3 that if this expression is to be less than 0·05, $(z_0 - t)$ must be at least 7 min for 8 common check-in positions or 2 min for 9 common check-in positions.

The second restriction raises similar difficulties, and these can be resolved in a similar way. For this condition to be satisfied, it can be shown that the transfer time must be at least 3·6 min before close-out.

These results may be summarized as follows:

(i) Under no circumstances should the transfer time be less than 3·6 min before close-out.

(ii) If the transfer time were between 3·6 and 7 min before close-out, 9 common check-in desks would be required.

(iii) For transfer times earlier than 7 min before close-out, the number of common desks required would be as given in Table 2.

The difficulties inherent in the queueing process at any restricted or final desk are more serious. In the first place the input is unusual. Examination of individual flight records collected during the field studies had shown that the number of passenger groups s, who would report to a final desk, out of 20 groups on a flight, would have the binomial distribution:

$$b(s, 20; 1-F_0) = \binom{20}{s}(1-F_0)^s F_0^{20-s}$$

where $F_0 = F(z_0, t)$.

Now every queueing process at a final or restricted desk starts from rest, as there is no queue at the moment z_0 when it starts. Furthermore its duration in terms of number of passenger groups served is quite short, so the process does not attain statistical equilibrium. Existing theory was inadequate to deal with such a problem and again Monte Carlo methods were used.

The check-in process at restricted or final desks was studied by simulating check-in for 100 flights under a variety of conditions. The basis of each simulation was:

(i) that $(z_0 - t) < 60$ min, and that the restricted desk would be opened at z_0 and closed at t.

(ii) that the number of passengers checked in at the restricted desk would have the binomial distribution $b(s, 20; 1-F_0)$ given above.

(iii) that there would be 1 or 2 clerks serving at each desk.

It was assumed further that the case $(z_0 - t) = 60$ min would correspond to a completely restricted check-in system, and that in this case all passengers arriving earlier than z, which would here be flight open rather than transfer, would disperse in the lounges and return at random during the period (z_0, t) to check in. Thus all 20 passenger groups would "arrive" as far as check-in was concerned during the 60 min period prior to close-out for any flight.

The results for the case $(z_0 - t) = 60$ min, which corresponds to a completely restricted check-in system, are given in Figure 4. This figure shows that for only one check-in clerk per flight the mean passenger waiting time

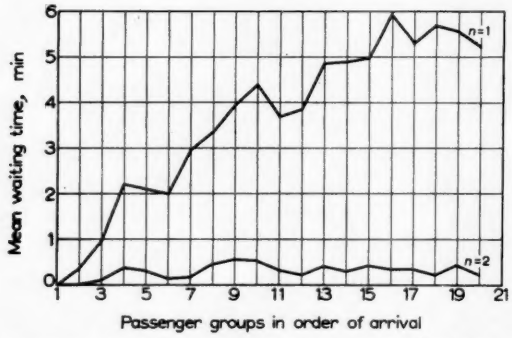

FIGURE 4. *Mean waiting time per passenger group at restricted desks (Monte Carlo results: $z_0 - t = 60$).*

would be completely unacceptable for groups later than the second, and suggested a further simulation for smaller values of $(z_0 - t)$ of about 5 min. It was deduced by this means that for 1 clerk, transfer should not take place more than 6 min before close-out.

With 2 clerks per flight however, Figure 4 shows that any transfer time is acceptable. Thus one can deduce the very important result that if the restriction were that mean passenger waiting time must be about 0·5 min, a completely restricted check-in system, with all passengers for a flight reporting during the hour preceding close-out, would be acceptable if each check-in desk were manned by two clerks.

The next question was whether 1 clerk per desk, for $(z_0 - t) = 6$ min, would produce acceptable completion times. Figure 5, also derived from

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the Monte Carlo calculations, shows the distribution of completion times for 100 flights when transfer was assumed to take place 6 min before close-out. Even allowing for the smallness of the sample, it is clear that completion of check-in would occur later than 5 min after close-out for more than the acceptable level of 5 flights in 100.

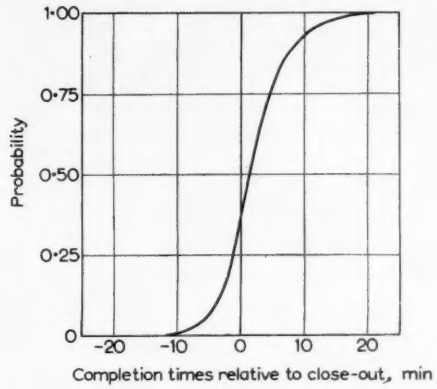

FIGURE 5. *Cumulative probability distribution of completion times at final desks for transfer 6 min before close-out* ($z_0 - t = 6$ *min*). (*Monte Carlo results.*)

Finally, would the provision of two clerks per restricted check-in desk result in acceptable completion times, and if so, for what range of transfer times? Figure 6 shows the probability of a completion time exceeding 5

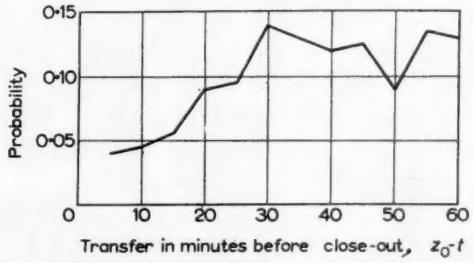

FIGURE 6. *Probability of late completion at final desks with two clerks (Monte Carlo results).*

min, as a function of transfer time. This suggests that two clerks would suffice, for transfer times not earlier than 13 min before close-out, but not otherwise. In fact it was decided to relax and accept 15 min.

On the basis of these results Table 3 was constructed to show how many clerks would be required to man the restricted check-in desks in a composite system, for different transfer times.

TABLE 3

NUMBER OF CHECK-IN CLERKS REQUIRED AT FINAL DESKS

(z_0-t) min	0	5	10	15	20*	40*	60*
No. of clerks	0	2	4	6	10	20	30

* 1 patrolling "booster" clerk is allowed for every two flights to satisfy completion time restrictions.

It was clear that even if all check-in was carried out at common desks, the traffic intensity, using a single sorting clerk, of the queueing process associated with ticket sorting would only be

$$\epsilon = \frac{360 \times 2{\cdot}1}{3600} = 0{\cdot}21$$

so that the delay in sorting tickets would be negligible. So long as any common check-in at all was to be carried out, however, one sorter would still be required.

The regular input to the addlisting process, and the upper limit to the service time of 6 min showed that at most two clerks would be needed for this task. Each clerk would deal with one flight at a time, and the clerks would take alternate flights. One clerk would suffice if the transfer time were so far ahead of close-out that the probability was small that the number of passengers to be addlisted was so large that the operation would take longer than the 5 min interval between consecutive flights.

The average number of coupons per flight batch, for a transfer time of z_0 is $30F(z_0, t)$ or $30F_0$. The mean addlisting time is then

$$\{(0{\cdot}05)(30)F_0 + 4{\cdot}5\} \text{ min} = 1{\cdot}5F_0 + 4{\cdot}5 \text{ min}$$

For this to be less than 5 min, $F_0 < 0{\cdot}3$ and so $(z_0 - t)$ must be not less than 46 min, as indicated by Figure 2. However, as the number of passengers to be addlisted per flight has the binomial distribution, the addlisting time must vary and a transfer time 46 min before close-out would produce excessive and (during the busy period) uncontrollable queueing.

The amount of interference associated with a probability of 0·05 that the time to addlist would exceed 5 min, though arbitrary, appeared

acceptable for practical purposes. Consequently it was necessary to determine z_0 so that

$$\text{Prob}\{(0{\cdot}05)(30)F_0 > 0{\cdot}5\} \leqslant 0{\cdot}05$$

This is the same thing as finding the maximum value of p satisfying the relation

$$I_p(10, 21) \leqslant 0{\cdot}05$$

By reference to Pearson's "Tables of the Incomplete Beta Function", it is found that p must not be greater than about 0·193. Hence, from Figure 2, $(z_0 - t)$ must be greater than 52 min.

Thus two addlist control clerks would be required except for transfer times earlier than 52 min before close-out, in which case one would suffice. Furthermore one such clerk would be required even if completely restricted check-in were in force, to carry out the baggage reconciliation procedure and transmission of data to Load Control (4·5 min per flight).

At this point in the investigation it was possible to obtain an estimate of the total number of check-in staff required by composite systems with various transfer times, and Figure 7 shows the results.

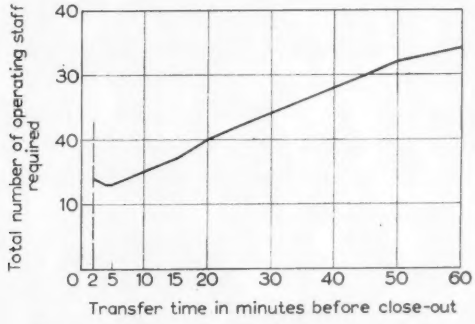

FIGURE 7. *Showing the optimum transfer time.*

It is difficult in practice, however, to operate a composite check-in system with transfer times which are not multiples of 5 min, so Figure 7 shows that the optimum check-in system for the hypothetical station represented by the model, would be a composite system with transfer taking place 5 min before close-out.

Conclusions

The importance of this study did not lie merely in the determination of the optimum system for a hypothetical station. In fact, the assumed traffic

loading at this hypothetical station corresponded very closely to the real situation at the two major stations at which the field work was carried out. The results of this study therefore could be and have been applied and appear in practice to fulfil their promise. It is hoped to validate the model quantitatively in the near future.

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The value of the study, however, did not lie only in its immediate usefulness. Its value also lay in the greater insight into the mechanism of check-in processes which it gave, and the consequent ability to resolve certain traditional problems associated with check-in systems. For example, it has been suggested from time to time that it might be possible to dispense with final desks and rely on a completely common check-in system at some stations. The study showed that this could only be done if the passenger arrival pattern fulfilled certain specific conditions. In practice it is found that these conditions are never met, so that final check-in desks are always necessary.

Furthermore, it is now possible to predict, with a fair degree of accuracy, the number of staff required to operate a check-in system which is still on the drawing board. Suppose for example, in a proposed terminal design, the speed of conveyors to and from the baggage sorting room for common desks were so low that the transfer time had to be at least 15 min before close-out, as against an optimum of 5. The slowness of the baggage conveyors would give rise to a need for more check-in staff than the potential minimum. The additional staff numbers needed, which could now be predicted, might be so great that the proposed baggage conveyor system would be unacceptable.

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Appendix

The approximation to the mean waiting time for a multiserver queueing process with Pearson III service-time distributions used in these studies, viz:

$$\bar{W}_n = \bar{E}_n\left(\frac{1+V^2}{2}\right)$$

was suggested by D. G. Owen, who used it in an unpublished investigation carried out at the British Iron and Steel Research Association some years ago, but did not test its accuracy empirically.

Results given by this formula have been compared with those derived by Monte Carlo simulations of several check-in queueing processes, and the two sets are given below.

TABLE 4

COMPARISON OF DIFFERENT ESTIMATES OF MEAN WAITING TIMES

n = No. of servers	*Station* A Formula	$b = 2{\cdot}50$ $V = 0{\cdot}73$ Monte Carlo	*Station* B Formula	$b = 1{\cdot}64$ $V = 0{\cdot}85$ Monte Carlo	*Station* C Formula	$b = 1{\cdot}59$ $V = 0.96$ Monte Carlo
7	—	—	2·59	2·25	1·70	1·64
8	—	—	0·48	(0·27)	0·40	(0·51)
9	—	—	0·17	0·15	0·17	0·12
10	21·8	(8·4)	0·07	0·07	0·06	0·05
11	1·09	0·92	0·03	0·04	0·03	0·03
12	0·38	0·45	—	—	—	—
13	0·16	0·16	—	—	—	—
14	0·07	0·07	—	—	—	—

The two sets of results are in fairly close agreement, and suggest that for the Pearson type III distributions of service times encountered in practice in check-in systems (for which, usually $0{\cdot}70 \leqslant V \leqslant 1{\cdot}0$) the decision whether to provide n or $n+1$ servers, on the basis of mean waiting times calculated from Erlang's formula and the factor $(1+V^2)/(2)$ is so clear cut that extensive simulations are unnecessary.

Abstracts and Reviews

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Scientific Programming in Business and Industry.
ANDREW VAZSONYI.
John Wiley and Sons, New York; Chapman and Hall, London, 1958. xix + 474 *pp.* 108*s.*

The application of mathematical methods to business problems is a young science and books on the subject are still fairly rare. Books at all levels, from the most elementary introduction to the highly refined monograph are, of course, a necessity in all fields of science. There is, however, a particular danger about elementary texts in the field of operational research. These texts, often described by the publishers as revealing the new methods to the non-mathematical businessman, can persuade him to try and apply the praised recipe to some of his problems; and if the results are disappointing, it is the new method that will be blamed, not the fact that an unschooled novice has tried to use a tool needing great expertise. It is an unfortunate fact that most introductory books on our subject are of the cookbook type. I am therefore very happy to report that the book under review is not.

It is written for the business executive with little or no mathematical background. But Vazsonyi does not pretend that it is possible to explain the application of mathematical techniques without mathematics. In the preface he advises his non-mathematical reader to get together with a mathematician friend, and I feel sure that by the time our non-mathematician has worked through the 460-odd pages he will have acquired sufficient mathematical understanding to see the necessity of expert mathematical help when applying programming to actual problems in his business, but he will also have gained a good idea of the capabilities as well as the limitations of mathematical programming.

For the same reasons the book is also an excellent first text for the mathematician intending to specialize in the subject. It has the drawback that, apart from a few footnotes, there are no references to other literature. In a book of this sort a bibliography to further reading is a must.

The book is subdivided into three parts and altogether 13 chapters. Part I deals with the fundamentals, i.e. an introductory description of the use of mathematical models in business and illustrative examples of linear programming.

Part II describes the basic mathematics of programming. Starting with an explanation of the simplex method it goes on to deal with the dual theorem and its uses. Then follows a chapter on the geometry of linear programming, the perspective diagrams of which will be helpful to people with visual imagination. There is a chapter each on convex programming, dynamic programming and elementary theory of games. Throughout this part the theoretical explanations are well linked to practical examples.

Part III called "Programming in Production and Inventory Control" has four chapters. The first, on statistical inventory control is the weakest of the whole book, mainly, I feel, because it is impossible to deal with this subject satisfactorily in less than 100 pages, even if one restricts oneself to the barest elements.

The last three chapters of the book, on various scheduling models, break new ground. Descriptions of these models were previously available only from some papers, mostly by the author himself, and published in a variety of American journals. They require a somewhat higher level of mathematical sophistication than the previous chapters. The business man who can work through these successfully can be trusted to have acquired a good knowledge of the use of mathematical models.

It is a great pity that so good a book is marred by an abundance of misprints in the text as well as in the formulae. Most of these are easily recognized, especially when one has a mathematician's help in reading, but the misprints in formulae (9), (10) and (12) on page 255, and 256, can easily lead to wrong results if the mistakes are not spotted.

A further point of criticism is that formulae and diagrams are numbered only within each chapter, and that there are no indications of the chapter number on top of the page. This forces one to consult the contents index when looking up one of the frequent references to formulae given in previous chapters.

Nevertheless this book is highly recommended.

E. K.

The Fundamentals of Statistical Reasoning. (No. 3 of Griffins' **Statistical Monographs and Courses.**)
M. H. QUENOUILLE.
Charles Griffin and Co., London, 1958. 169 *pp.* 24*s.*
The author sets out to give a fuller discussion of statistical principles than that to be found in many textbooks. He observes that these are

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often concerned primarily with either a descriptive treatment of standard methods or with the mathematical examination of their properties, consequently the reasons underlying choice of methods tend to receive too little attention. The first two chapters give a rapid survey of some of the ideas of probability and distribution theory; this space might have been better used to relieve the compressed treatment in later chapters. Chapters 3 to 5 deal respectively with properties of estimators, with tests of significance and with the maximum likelihood principle. Chapter 6 contains a discussion of fiducial probability and fiducial limits. The subject-matter is naturally controversial, but the treatment should be of interest to those who do not accept the fiducial approach. The final chapter opens with some interesting comparisons of power curves of different tests of the same hypothesis. It goes on to discuss desirable properties of tests and the general likelihood-ratio test; it includes a brief outline of Wald's decision function approach.

The necessarily condensed treatment of a book of this size makes it unsuitable for readers who do not already possess a working knowledge of the methods of statistics. Given this background there is much of interest, although the presentation of inductive principles is exclusively in terms of fiducial probability. The full discussion of the power of a test of significance should prove a useful corrective to the tendency to regard the control of Type I errors as the only purpose of such tests. Many readers will find the numerous concrete examples, often discussed in numerical terms, preferable to the entirely abstract approach so often encountered in modern statistical writing.

J. H. C.

Studies in the Mathematical Theory of Inventory and Production.
K. J. ARROW, S. KARLIN and H. SCARF.
Stanford University Press, Kentucky, 1958. x+340 *pp.* $8.75.

This mathematical treatise is an indication that operations research (and in particular, inventory theory) is out of the adolescent phase. While the volume is not a handbook (it makes no pretence of being one), the authors have presented a thorough mathematical analysis of inventory problems as a class of optimization problems. This is not a book for the novice in operations research or the informed manager but the operations research practitioner should find it an invaluable reference work. The first chapter

of the book is required reading for all concerned with operations research and inventory theory. Professor Arrow's introduction discusses inventory in terms of Keyne's analysis of money as a commodity. The motives for holding cash, i.e. transactions, precaution and speculation are similar, if not identical, to those for holding inventory. Chapter 2 (The Nature and Structure of Inventory Problems) is a clear and concise report of the structure of inventory models, the important factors and the various types of analysis. This chapter should be required reading for students and practitioners. Chapter 3 is a summary of the approaches and the results obtained in the technical sections. The above three chapters comprise Part I. Parts II, III and IV (Optimal Policies in Deterministic Inventory Policies, Optimal Policies in Stochastic Inventory Policies, Operating Characteristics of Inventory Policies) each consist of a series of chapters under the main subject heading. The chapters would stand alone. Together they present a more complete picture of the present state of inventory theory. Chapter 9 is devoted to an analysis of the Arrow–Harris–Marschak dynamic inventory model. Chapter 10 extends the model to include lead times. These two chapters are an important contribution to inventory theory. Part IV discusses simple policies and the methods for selecting optional policies from a group of possible policies. Inventory practice as typified by most present investigations, follows this practice. Simple policies are preferred because they result in simple procedures; they allow predictions of future operating characteristics and they are often optional under many sets of plausible and realistic conditions. The chapter deals with a number of problems of a stochastic nature in terms of steady state solutions. (*sS*) policies are discussed in considerable detail. One chapter (Chap. 11) is devoted to optimal policies for hydroelectric operations, which falls within the scope of the volume. Water, like money, can be treated as a commodity subject to inventory fluctuations. While the presentations are mathematical and written at a high level of sophistication, the considerations and assumptions are not at all unrealistic. The practitioner might have preferred some practical examples, but the volume is not intended as a collection of case studies. The authors, with admirable support by Messrs. M. J. Beckmân, John Gessford and Richard F. Muth, have written an important and original work.

E. K.

Principles and Applications of Random Noise Theory.
JULIUS S. BENDAT.
John Wiley and Sons, New York; Chapman and Hall, London, 1958. 431 *pp.* 88*s.*

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This book was written primarily for engineers and opens with three chapters introducing the concepts of stationary random processes, auto and cross-correlation and power spectra. They also include a summary of the probability theory needed in random noise analysis. The treatment does not demand a wide mathematical background and is designed to be illuminating rather than rigorous. Chapter 5 presents empirical and theoretical arguments for the wide applicability of an exponential–cosine auto-correlation function. Chapter 10 deals with the distribution of zeros of a fixed sine wave or of a sine wave process accompanied by random noise. The rest of the book is concerned with applications of the theory to electrical engineering problems. Chapters 4 and 9 deal with the characteristics of optimum filtering systems and extend the work of Wiener in this field. Chapter 6 considers analogue methods of examining the effect of a given filtering system. Chapters 7 and 8 discuss the measurement of auto and cross-correlation and the effect of such factors as correlator bandwidth and smoothing time.

Much of the subject matter of later chapters is relatively new. It demands a knowledge of Laplace transforms and of circuit theory and is likely to be of interest mainly to the specialist. The introductory part, being concerned with ideas of greater generality, will be of more interest to operational research workers. This work is well presented, and the author's claim that no previous knowledge of probability theory or Fourier integrals is needed is justified. The book should thus prove a useful introduction to the subject of random noise for a wide variety of readers.

J. H. C.

Operational Research (Betriebliche Verfahrensuntersuchungen).
E. LIEBEL.
Ergmetall, **11**, 438–446 (September, 1958). (In German.)

The development of operational research from the war years to recent industrial applications is outlined, and two examples connected with the mining industry are considered. The first of these sets the problems connected with the transport of ore from five hypothetical ore preparation plants in different parts of the world to an iron works in Germany, while the second works out in detail a complex problem concerned with the

transport of stone from a works to three plants for treatment, and its subsequent dispatch to five receiving areas. The method of applying operational research techniques to the problem, including the use of calculating machines is given, together with the costs of the method (29 references).

Operational Problems Requiring Documentation Research.
VERNER W. CLAPP.
NATO Advisory Group for Aeronautical Research and Development, Report 48 (Feb. 1956).
The operations of documentation are so inter-related that no part of the cycle of treatment, storage, retrieval and the creation of a new document can be isolated, especially the present habits of information gatherers as these assume existing methods of documentation. Most enterprises show special activity in one or more parts of the cycle while assuming stability in other parts, and as immediate needs are served, no broad assignment for criticism of the whole cycle is present. The Dewey decimal classification is discussed from the point of view of the information that might have been obtained at the outset which might have obviated the 20 years' work needed to solve its problems so far. The general features of information systems, such as the citation of documents, are further discussed.

Linear Programming. Study of the Variation of All Parameters. Method of Sequential Solution.
MARCEL COURTILLOT.
C.R. Acad. Sci., Paris, **247**, 670–673 (1958).
Mathematical. In the expression

$$\sum_{i=1}^{i=n} a_{ij} x_i T b_j, \quad x_i \geqslant 0, \quad i \epsilon (1, \ldots, n)$$

where T is $\leqslant$ or $\geqslant$ and $j \epsilon (1, \ldots, m)$, introducing variables $x_{n+1}, \ldots x_{n+m}$ representing an application of a vectorial space U_{m+n} in a vectorial space V_m maxima are obtained.

Variations of b_j, c_i, a_{ij} are then separately formulated.

The sequential method is then used.

If S_m is a simple matrix relative to the base of the maximum of a problem P_m with m constraints and if a supplementary constraint is added

$$\sum_{i=1}^{n} a_{im+1} x_i T b_{m+1}$$

where T is as above, the matrix for the maximum S_{m+1} is worked out.

News and Notes

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WITH this issue the *Operational Research Quarterly* enters its tenth year of publication. Those of our readers who glance at the cover will have noticed that certain changes have been made during the past year, in particular that Pergamon Press Ltd. have undertaken the publication of the journal since September. Those who start by studying the cover will have noticed a further change with this issue. Max Davies has resigned as Joint Editor.

It has not been the editors' custom to give expression to their views either overtly or anonymously, but perhaps I may be allowed to break this rule in order to say a personal word of farewell on this occasion. Max Davies has been responsible with me as Joint Editor since the *Quarterly* was first published. When it all began he comforted himself (and me) with the conviction that it could not possibly survive more than a couple of years—one of his very rare errors of judgment. The way in which the circulation has steadily built up is a tribute to his enthusiastic success in proving himself wrong; it has also meant a relentlessly increasing burden of work for him. He has set a high standard for our new publishers to emulate. Thank you, Max.

May I, too, thank Peter Myers for all the work he has put in for us in the United States. I hope that now he is relieved of this chore he will find time to pay us a long postponed visit in England. I also particularly thank Colonel and Mrs. Taylor and Mrs. Kirkham, who are probably not even names to most readers but have done yeoman work in handling correspondence and subscriptions.

The Executive Committee of the Operational Research Society have now established an Editorial Committee. They have very kindly appointed me Chairman of this committee and my colleagues are Messrs. R. H. COLLCUTT, S. L. COOK, Professor P. M. MORSE, Dr. R. J. SMEED and Mr. H. R. WATKINS. I look forward to further development of the *Quarterly* in association with them and our publishers.

R. T. EDDISON

McKinsey Foundation Awards

ANNUAL awards, in the amount of $1,000 for the first prize and $500 for the second prize, will be presented to the authors of the two best articles appearing in each of the following journals: *The Harvard Business Review*, published by The Graduate School of Business Administration, Harvard University; the *Journal of Business*, published by the University of Chicago; and the new publication of the University of California, *The California Management Review*.

Particulars may be obtained from: Miss Joan Keene, McKinsey Foundation for Management Research, 60 East 42nd Street, New York, New York.

2nd International Conference on Operational Research

THE second International Conference on Operational Research organized by the International Federation of Operational Research Societies (IFORS) will be held in Aix-en-Provence, France, from 5 to 10 September, 1960.

The Programme Committee would welcome suggestions for papers (or groups of papers) to be presented at the Conference. Suggestions should be sent to the Secretary of IFORS, 11 Park Lane, London W.1, England, before 1 May this year, with a copy to the Secretary of the Operational Research Society of the country of origin. Manuscripts will be required by 1 December, 1959, in order that preprints can be made available before the Conference.

IFORS came into existence on 1 January this year, having as its objects "the development of operational research as a unified science and its advancement in all nations of the world". The first international conference on the subject was held at Oxford in 1957.

Canadian Operational Research Society

THE Canadian Operational Research Society (CORS) was formed in Toronto in April 1958. Omond M. Solandt is President and a Council has been elected to serve until the first annual meeting. Peter J. Sandiford of Trans-Canada Air Lines, Montreal, is Secretary.

Membership consists of Technical Members and Associates; the former are those who are actively engaged professionally in operational research;

Associate membership is available to persons employed in management, education, etc., but all members are enlisted as Associates initially.

The object of the CORS is the advancement of the theory and practice of operational research in Canada by:

(1) Holding conferences for the presentation of technical papers on original researches and providing other means for the exchange and dissemination of information on operational research.

(2) Stimulating contact between workers engaged in operational research.

(3) Fostering development of local sections and supplying guidance and assistance to these sections.

(4) Promoting education and training in operational research.

(5) Establishing and maintaining liaison with operational research groups in other countries and arranging Canadian representation at international conferences on operational research.

The Society will encourage the establishment of Sections of the Society in local areas, and groups engaged in operational research, management sciences or related fields will be eligible.

The first annual meeting of the Society will be held at Room 102, Mechanical Engineering Building of the University of Toronto on 7–8 May, 1959. Those wishing to attend should write to Dr. A. E. Paull, Abitibi Power and Paper Co. Ltd., 403 University Avenue, Toronto, Ontario

Operational Research in the Midlands

It is suggested that an Operational Research Study Group for the Midland area, but not necessarily based in one town, might be a useful means of contact and discussion among operational research people, and as a means of promoting interest in operational research in industry and other bodies. Would those interested get in touch with: W. N. Jessop, Operational Research Section, Courtaulds Limited, Coventry.

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A Method of Fixing Desirable Stock Levels, and of Stock Control*

R. H. COLLCUTT, J. BANBURY, R. G. MASSEY and R. A. WARD
Operational Research Department, The British Iron and Steel Research Association

The size of the stock of raw material kept by a company in the steel industry was such that it could not be kept in the main stockyard but had to be spread over two sites. This had an adverse effect on the cost of stockholding. Analysis showed that the stock was large because of the lack of a systematic method of stock control.

A study of works records indicated the form that a stock control system should take, using all the available information concerning the present stock, the future production, and a forecast of what raw material was still likely to be delivered as a result of orders placed in the immediate past but as yet not completely fulfilled.

A "simulation" technique, which is fully described in the paper, was used to determine the effect of using the method of stock control which had been devised. The results indicated that the stock could be reduced to about half its original size, so releasing about £100,000 of working capital and, by concentrating it in one stockyard, reducing the costs of transporting and handling.

* One frequently sees in the literature of operational research that the aim of a study should be to find the "optimum" solution to a management problem; this may be a laudable aim in theory, but it is my experience that it is rarely achieved in practice. There are many reasons why this should be so, of which perhaps the three most important are:

(i) To wait until the optimum answer has been derived may often render the study obsolete.

(ii) Some of the value judgments necessary will frequently not allow of an absolute assessment of the optimum.

(iii) The optimum is in many cases not well defined.

In these circumstances operational research achieves a useful result if it can specify a "better" solution to the problem than has been available hitherto. I believe that stock control problems are usually in this category. This case study concerns an occasion when the Operational Research Department decided to follow a policy of aiming at a "better" answer without worrying about whether it was in fact the "best".

R. H. COLLCUTT

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1. Introduction

1.1. This paper describes an investigation for a particular Company, into the feasibility of reducing their stock of raw material. The methods used in the analysis may be applicable to similar problems at other works.

1.2. This particular problem arose because of a desire to reduce the amount of the charges that were being incurred to the British Transport Commission. These charges arose primarily because the stock which it had been customary to keep could not be contained entirely at the works where it was required and part of it (about 50 per cent, on average) had to be kept on the other side of a British Railways main line; inter-site transport was by rail, and the B.T.C. levied a charge on all the loaded wagons crossing their tracks. In addition, the capacity of the Company's wagon unloading facilities, which had to be shared between the two sites, was such that it could not cope with peaks in the deliveries of raw material; as a result, standage charges were high.

1.3. The aim was therefore to see if a method could be found of reducing the amount of stock so that it could be contained entirely at the works where it was used, and to do this without materially increasing the chance of runout. The problem of standage and the costs of handling the stock were made the subject of a separate study.

1.4. It was decided first to study the existing method of stock control, and to try and discover what were the important factors which affected the stock level.

2. The Existing Stock

2.1. *The Ordering Procedure and the Mode of Delivery by the Supplier*

2.1.1. The Company had an agreement with two main steel suppliers that a certain quantity of steel would be bought from them on an annual basis; however, no steel was delivered to the Company's works until, in addition to the general agreement, an order for a specific amount had been placed. It was the policy to place such an order on the Wednesday of each week, and the quantity ordered was to all intents and purposes constant, the only variable being the steel qualities asked for. Occasionally, however, it was claimed that emergency changes in the size of the demand were made when the management became alarmed because the stock was either too large or too small; then either orders were cancelled or extra supplies sought from other suppliers.

2.1.2. There were two main suppliers (A and B) and the Company intended in the future to limit their custom more or less entirely to those two firms. The size of an order was defined in terms of complete casts of steel, the average tonnage of which was different for the two suppliers.

2.1.3. The response of a supplier to a demand for steel varied from time to time and examination of Company records indicated that supplies would arrive in batches over the following 4 weeks. Table 1 shows the average response to a demand placed on the two suppliers. Table 2 at the end of the paper shows a sample of responses for demands for individual casts over a period of 9 months.

TABLE 1

AVERAGE RESPONSE TO A DEMAND FOR STEEL

Supplier	Proportion by Weight delivered in:				
	Current Week (per cent)	1st Week (per cent)	2nd Week (per cent)	3rd Week (per cent)	4th Week (per cent)
A	5·5	50·6	26·1	13·7	4·1
B	3·8	49·5	37·4	7·5	1·8

2.1.4. There were occasions when a demand for a cast was never fulfilled; in practice the Company would know that this was going to occur within a week or two of the order being placed. Taking account of this the average response to a demand for a cast from supplier A was 56 tons and from supplier B was 109 tons. The averages in Table 1 include the cases when the response was nil.

2.2. *The Demands of Production*

2.2.1. Analysis of 18 months' records showed that the weekly production requirements for raw material varied considerably; see Figure 1. As far as could be ascertained, the production in any one week had very little correlation with the production in any other week.

2.3. *The Existing Stock Level*

2.3.1. The level of stock at any time depends upon the orders for steel placed in the past, the response of the suppliers, and the demands of production. From first principles it might be expected that a constant weekly order for stock of the type placed by the Company would in the

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circumstances result in violent fluctuations in the level of stock from time to time. These fluctuations might be accentuated by the more or less intuitive decisions to increase or cut back on stock. Figure 2 shows the variations in stock which accrued during a period of 18 months; the minimum and maximum amounts were 2,000 tons and 9,000 tons with an average of 5,300 tons. Figure 2 also shows the standage charges for railway wagons incurred during the same period; there was a marked rise in the charges during stockpiling—that is, during peak deliveries of steel.

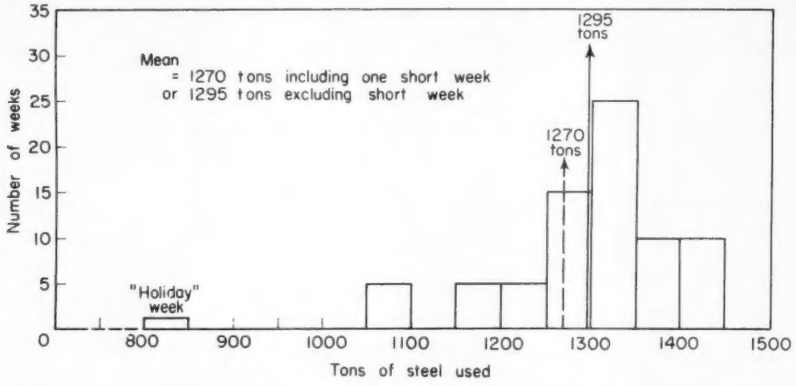

FIGURE 1. *Distribution of tons of steel per week for a period of* 18 *months* (16 *shift weeks and* 1 *holiday week only*).

2.3.2. Although this type of variation in stock level was expected, it was not known whether the actual size of the variation had been affected by any sudden decisions to increase or decrease orders. It was therefore decided to simulate the process, assuming constant size of order, so as to check the variation to be expected and to demonstrate to management the effects of their existing re-order policy.

2.4. *Simulating Current Procedures*

2.4.1. It would have been quite unrealistic to attempt to derive stock levels merely by considering average deliveries and average production, since the variations in delivery and demand must cause a considerable part of the wide fluctuations in stock level. A method of analysis had therefore to be used which took account of these week-by-week variations, so that the coincidental occurrence of the adverse effects which caused the stock level fluctuations (large delivery—low production, small delivery—

high production), would have a chance to arise in the same way as they might in practice.

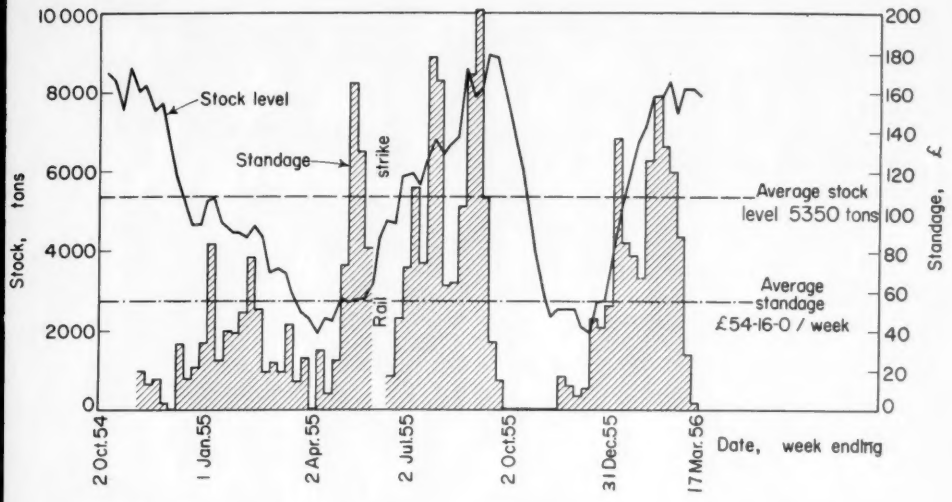

FIGURE 2. *Billet stock level and standage paid for the period October* 1954–*March* 1956.

2.4.2. The method of simulation takes account of these variations and, in effect, involves operating—on paper—the system being studied, on the basis of past experience. For example, if it can be shown (as it was in this instance) that the pattern of delivery of one cast of steel was independent of the pattern of delivery of another cast ordered in the same week, then the total pattern of delivery of a complete week's order could be simulated by selecting at random the patterns of delivery shown in Table 2, of sufficient casts to make up a week's order. A running total could then be kept of the deliveries of steel in the weeks following the placing of the order. If this process were repeated again and again, a picture could be built up of the deliveries which would take place week-by-week if a constant weekly order were placed.

2.4.3. The fluctuations of stock level were simulated by selecting at random from the distribution of the weekly production demands for steel, and subtracting these values from the simulated, week-by-week, deliveries. This is permissible only if it can be shown (again, as it was in this instance) that:

(a) the amount produced in one week is not affected by the amount

produced in the preceding and succeeding weeks; and

(b) the amount produced in a particular week is not affected by the amount delivered in that week.

2.4.4. In practice a suitable weekly rate of ordering had first to be derived. The average demand for raw material was 1,270 tons a week and, in order to avoid long-term building up or running down of stock, the weekly order was made approximately equal to that amount. This requirement could be satisfied by ordering each week:

7 casts from supplier A	= 392 tons
and	
8 casts from supplier B	= 872 tons
Total	1,264 tons/week

The resulting 6 tons difference between intake and demand could be made up by ordering an extra cast from supplier A once every 9 weeks.

2.4.5. By repeatedly selecting this number of casts from each supplier at random from the lists given in Table 2, and by keeping running totals, the week-by-week deliveries over a long period were simulated. The resulting distribution of these weekly deliveries is shown in Figure 3, which shows that the variation was between 800 and 2,000 tons.

2.4.6. The weekly production requirements were then subtracted from these deliveries (as described in paragraph 2.4.3) to give the stock fluctuations. These are shown against a time scale in Figure 4. It will be seen that the stock could be expected to vary between 2,000 and 10,000 tons, with an average of 5,650 tons. This result bore a striking resemblance to what was happening in practice, and demonstrated that the stock level under these conditions could not be confined between narrow limits if a constant amount of steel was ordered each week.

2.4.7. It was now necessary to consider how the weekly order could be varied to limit the fluctuations in stock level.

3. The Proposed Method of Stock Control

3.1.1The previous simulation had indicated the need to vary the weekly order for steel to take account, as far as possible, of:

(a) the present stock level;

(b) the variations in delivery;

(c) the variations in the production demand.

The level of stock should be fixed so that only very occasionally would it be necessary to change the planned production programme because the

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TABLE 2A

ARRIVALS AFTER ORDERING ONE CAST FROM SUPPLIER "A"

(expressed as percentage of the average cast)

Total Tons in Cast	Arrivals (percentage of 73 tons)				
	In Week of Order	Week after Order: 1st	2nd	3rd	4th
65		7	82		
84		100	15		
89		106	16		
33		45			
33		45			
30		41			
83			113		
85			96		20
89		114		8	
59		81			
62				86	
79		94	15		
70			96		
49		35			33
23				26	6
82		112			
72	99				
79		108			
70		64	32		
79		28	81		
80	99		11		
68			82	11	
81		38	73		
84		115			
75		103			
80		87	24		
79		99	10		
85			115		
83			113		
72		99			
83			94	21	
81				101	9
76				103	
81				72	40
89		102	21		
87	64	57			
126		98		74	
74		90	13		
84				103	13
42		36	22		
131		73	109		
89		92	30		
81		111			
89		68			55
90		124			
10		14			
100		117		20	
66			91		
16			22		
54		74			
96	11	120			
57				78	
47		36		29	
68		48	38		7
	17 casts ordered produced no response				
Tons	Average response from casts which were delivered: 4	37	19	10	3	Total 73
Tons	Average response to all casts ordered: 3	28	15	8	2	Total 56
	Percentage response each week (average): 5·5	50·6	26·1	13·7	4·1	

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TABLE 2B

ARRIVALS AFTER ORDERING ONE CAST FROM SUPPLIER "B"

(expressed as percentage of the average cast)

Frequency of this Response	Total Tons in Cast	Arrivals (percentage of 116 tons)					Frequency of this Response	Total Tons in Cast	Arrivals (percentage of 116 tons)				
		In Week of Order	Week after Order:						In Week of Order	Week after Order:			
			1st	2nd	3rd	4th				1st	2nd	3rd	4th
2	83		28	44			1	135		115	2		
1	129		75		36		1	134		51	65		
1	115		99				1	7		6			
1	135		116				1	123			100	7	
1	129		106	5			1	108		9	84		
1	6		5				1	111		62	34		
5	111	60	35				1	117		68	33		
3	193		69	89	6	3	1	122		105			
2	128		29	76	5		1	121					105
2	3		3				2	134		71	44		
1	127		85	24			1	63		40	14		
2	117		65	36			1	106		91			
1	155		29	105			1	115		81	5	12	
2	119		103				1	123		106			
1	135		103			13	1	134			116		
2	99	61	24				3	107		3	90		
3	150		40	90			2	133		35	81		
1	30			26			1	136		90	27		
3	121		8	31	66		2	136		4	41	71	
1	136			117			1	75		65			
3	143		40	83			1	107			93		
2	101		67	20			1	58		50			
1	125		95	13			1	175			135	17	
3	112		46	16	34		1	142		7	116		
3	119		100	3			2	127		76	34		
3	116		73	27			1	11		10			
3	125		81	26			1	112			96		
3	90		42	31	5		1	142			123		
1	77		47	10			1	119			103		
2	139		25	95			2	93		23		57	
3	132		102	36			1	146		54	71		
1	149		97	31			1	74		64			
1	102	31			56		1	65		56			
1	133		54	60			1	134			116		
1	137		60	58			1	148		128			
3	123		27	71	8		1	153		14		118	
1	140		113	8			1	121		58	46		
1	87		75				1	133		71	41		
1	194			75	100		1	145		67	57		
1	65			58			1	33		28			
1	111	20	69	7			1	156		32	72	30	
1	102		88				1	52		33	12		
1	95		73	9			2	126		73	35		
1	119		65	39			1	97		55	28		
1	78				28	40	1	68		58			
1	127		39			70	1	137		117			

12 casts ordered were not delivered

Average response **excluding** orders which produced no delivery = 116 tons

Week of Order	Week after Order:				Total
	1st	2nd	3rd	4th	
4	57	44	9	2	116 tons
3·6	49·3	37·8	7·6	1·7	100 per cent

Average response including orders which produced no delivery = 107 tons

Week of Order	Week after Order:				Total
	1st	2nd	3rd	4th	
4	53	40	8	2	107 tons
3·8	49·5	37·4	7·5	1·8	100 per cent

steel required for it was not in stock.

3.1.2. It has been mentioned that the Company used a wide range of steels; it would therefore be necessary to ensure that the method of control devised should apply to each grade. For this purpose, the quality range was broken down into four groups defined according to the percentage carbon.

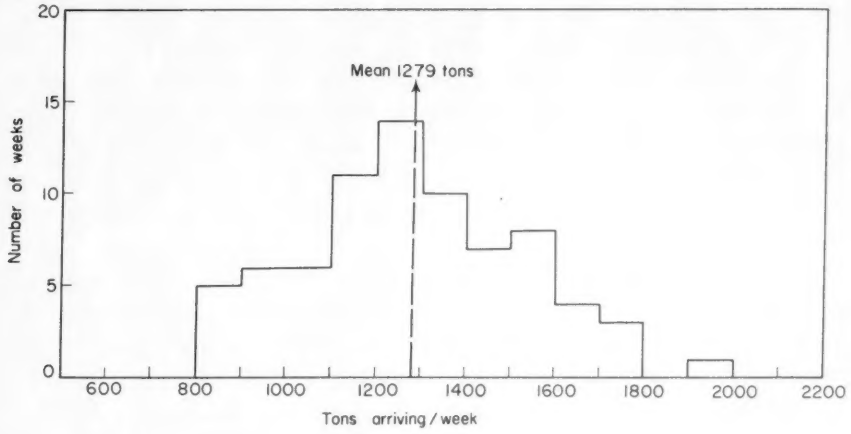

FIGURE 3. *Distribution of weekly arrivals of billets based on simulation experiment with constant weekly orders (described in section* 2.2.2).

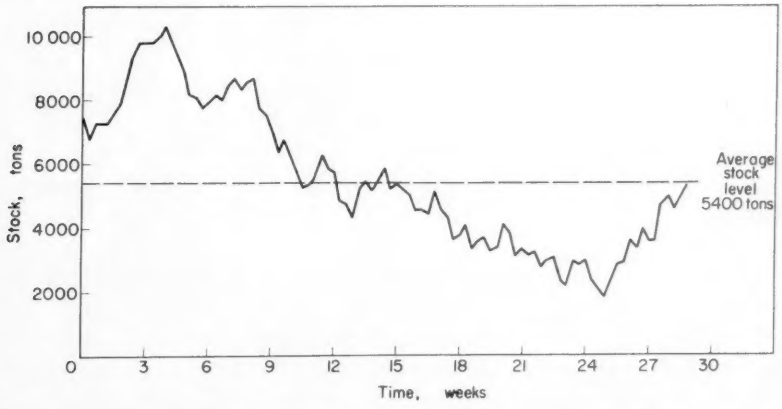

FIGURE 4. *Stock level derived from experiment described in section* 2.2.2 (*30-week sample constant weekly order*).

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3.1.3. A suitable stock control system would therefore make the order for this steel this week depend upon:

(a) the stock of this steel at present on hand;
(b) the production demands for this steel in the immediate future;
(c) the amount of this steel which could be expected to be delivered in the immediate future, as a result of orders placed in the past;
(d) the amount of this steel which could be expected to be delivered in the immediate future as a result of placing an order now.

3.1.4. The Company in fact kept a continuous record of what steel was in stock, and how much of the steel ordered from the suppliers in the immediate past was still outstanding. Production programmes were prepared from the order book for several weeks in advance and in fact a precise estimate could be made of the steel requirements of production in the succeeding 2 weeks.

3.1.5. Data of the type illustrated in Table 2 gave some measure of the response of the suppliers to orders placed upon them. So that information about the outstanding orders could be used in the control of stock, it was assumed that each order for a cast would be satisfied in due course by the delivery of a cast of the average weight given by the data in Table 2. Then any delivery of a part of a cast which had already been made due to a previous order could be subtracted from this average, and the balance used to forecast possible future deliveries of the outstanding part of a cast.

3.1.6. It was decided therefore to define a "normal" order for each of the four ranges of steel quality as the average consumption of each range. This was calculated from past consumption.

3.1.7. The information available on stock level, outstanding orders and future production was then used to decide whether the situation justified a deviation from the "normal" order for each range of steel in a given week, in order to forestall either a buildup of stock or a slowing down of production.

3.2. *Deciding when to vary the "Normal" Order for a Particular Grade of Steel*

3.2.1. Let P_0, P_1, and P_2 be the tons of this steel which are expected to be used in the remainder of the week in which the order is placed, in the next week and the week after, respectively. Then the total production demand expected up to the end of the next two complete weeks is $P_0+P_1+P_2$ tons.

3.2.2. Let T_0 be the tons of this steel in stock at the time of placing the order.

3.2.3. Let E be the total tons of this steel which can be expected in the future such that

$$E = e_0 + e_1 + e_{-1} + e_{-2} \quad (1)$$

where: e_0 = the average tons of this steel which may be expected at some time as the result of a "normal" order being placed this week;

e_1 = the average tons of this steel which may be expected at some time as the result of a "normal" order being placed next week;

e_{-1} = the average tons of this steel which was expected from the order placed last week, less any deliveries to date;

e_{-2} = the average tons of this steel which was expected from the order placed the week before last, less any deliveries to date.

3.2.4. Orders placed more than 3 weeks previously were ignored as if the total cast had not arrived any balance was unpredictable and small.

3.2.5. It is possible for e_{-1}, e_{-2} to be negative due to a more than average cast having already been delivered. In this case they were ignored.

3.2.6. The estimate of E thus obtained was a measure of the total deliveries which might be expected at some time in the future. However, as production could be forecast accurately only over the next 2 weeks it was considered necessary to safeguard production by ensuring adequate stock only over this period; production in subsequent weeks would be taken care of by orders placed at a later date. It was discovered that if E were calculated in the way outlined above, then nine times out of ten more than $\frac{1}{2}E$ tons would be delivered at the works in the following 2 weeks.

3.2.7. Thus the stock which could be considered available for production in the next 2 complete weeks could be fairly safely considered as T_0 (the amount in stock) plus $\frac{1}{2}E$ (the amount expected in the next 2 weeks).

3.2.8. However, if $P_0 + P_1 + P_2 > T_0 + \frac{1}{2}E$ (that is, if the production planned was greater than the present stock plus the arrivals expected in the period), then there would be a chance of more than one in ten that production might have to be curtailed due to a shortage of steel. It was decided that in this situation the "normal" order to be placed this week should be increased so as to cover the deficit calculated from

$$(P_0+P_1+P_2-T_0-\tfrac{1}{2}E).$$

3.2.9. As, however, an order placed this week had quite a good chance of not being delivered in full in the next 2 critical weeks, this meant that increasing the "normal" order by only the amount of the deficit would still result in a fair chance of a shortage developing. If the "normal" order were increased by twice the deficit, however, then three times out of four enough would have arrived in the next 2 weeks to cover the shortage. It was decided to use this rule; it should be borne in mind that, for the second week's production, a further week remained when another order could be placed to restore the balance if need be. Orders were to be placed weekly although each time account was being taken of a period a fortnight ahead. These rules were expected to reduce the risk of runout to very small proportions.

3.2.10. It was decided that if $P_0+P_1+P_2<T_0+\frac{1}{2}E$, then a "normal" order would be placed that week.

3.2.11. If this procedure were strictly followed, then there would be a tendency, over a long period, to build up stocks. Thus when the stock was at some predetermined level, action, such as not placing an order that week, would have to be taken. The selection of the "no order" level was made during the subsequent simulation using the rules outlined above, so that there was very little chance that the available stocking area would ever be full, and steel would be unlikely to have to wait in wagons and incur extra charges before being unloaded. The capacity of the stockyard alongside the works was 4,500 tons.

3.3. *Simulating the Effect of the Proposed Rules of Ordering*

3.3.1. The effect of making use of these ordering rules was simulated in a way similar to that described earlier. Initially, the technique was used to determine the "no order" levels by a process of trial and error—to which this method of analysis readily lends itself.

3.3.2. It was first necessary to simulate a production programme for each range of steel qualities, based on the records of past production. These records indicated that the production of each grade varied at random and no evidence was found of sequences of periods of high or low production. It was thus possible to reproduce a typical production programme for each steel by first randomly selecting a figure for the total week's production from Figure 1. The manner in which this total production was shared between the four steel grades was then decided by selecting

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at random (both between and within) from 3 of the 4 distributions of production by steel grade. The production of the fourth then made up the balance.

3.3.3. The simulation of this method of stock control was continued for a period equivalent to a year's operation of the works, and the resultant total stock level is shown in Figure 5. It will be seen that the stock remained well below the 4,500-ton limit set by the capacity of the stockyard. During the experiment the stock in fact never exceeded 3,100 tons; however, it is conceivable that under particularly adverse conditions it might exceed this figure. It was clear that the method of control devised was effective.

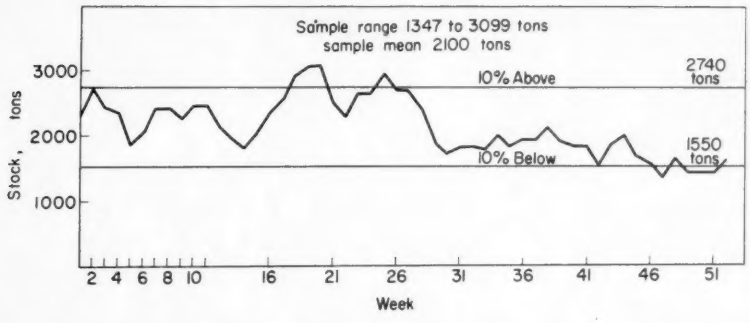

FIGURE 5. *Stock at beginning of each week of experiment described in section* 3 (*52-week sample variable weekly order*).

3.3.4. During the experiment the stock position never became so low that production had actually to be curtailed; however, four times the intended production with a particular grade of steel had to be postponed for a week and the slack taken up by bringing forward another order requiring another grade of steel. This amount of dislocation of the production programme was acceptable to the Company.

3.3.5. The Company were apprehensive that a method of control which deliberately varied the size of the demand on the suppliers from week to week might be unacceptable to the suppliers. Even under the existing system variation in the order of a particular grade of steel was considerable although the total quantity remained steady from week to week. In fact, it appeared that the variation in the size of weekly order would not be so very large; Figure 6 shows that in the experiment the order exceeded 1,870 tons 10 per cent of the time and fell below 1,170 tons 10 per cent of the time.

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3.3.6. A method was derived for calculating the "no order" level for a particular grade of steel from its rate of consumption in the works. As a basis for simulation, this level was first fixed so that nine times out of ten the week's production could be satisfied from stock; it was then modified by trial and error as a result of the experience gained during the simulation. The results are shown in Figure 7. It will be seen that when the steel forms a large proportion of the total throughput, the "no order" level is proportionately less than when the converse is true. However, the method of derivation became inaccurate when the throughput rate was small. In any case, under these circumstances it is doubtful whether a permanent stock is justified as an alternative to placing an order for raw material on receipt of an order from a customer for this particular grade.

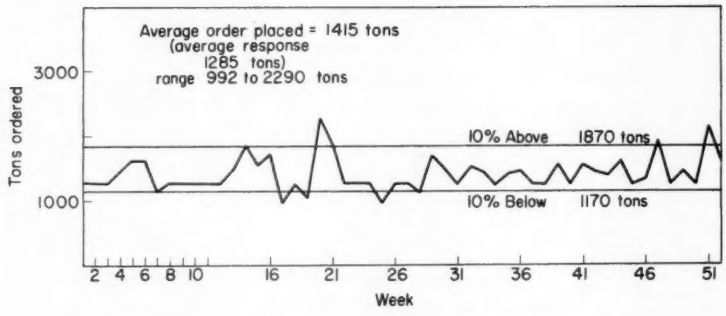

FIGURE 6. *Order placed each week of experiment described in section* 3 *(52-week sample).*

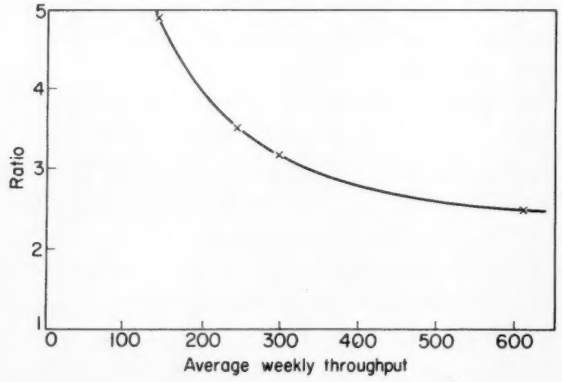

FIGURE 7. *The ratio "no order" level/average weekly throughput for various weekly throughputs (as used in experiment described in section* 3*).*

4. Conclusions

4.1. *Recommendations made to the Company*

4.1.1. By adopting a method of stock control which made use of information about

(a) the present stock,

(b) the planned future production programme, and

(c) the deliveries of steel which could still be expected as a result of orders placed previously,

the level of stock could be confined within narrow limits. The information required was already collected by the works as a matter of routine.

4.1.2. The method demonstrated would be suitable for these particular circumstances, and would be relatively simple to operate. Though it would not necessarily produce the level of stock which was the most economic (to find the "best" level, it would be necessary to assess the cost of having to change production programme) it would, however, make it possible to effect a considerable reduction in stocks. The changes in production which it would be necessary to make because of a lack of the required grade of steel would be negligible.

4.1.3. A decision should be made as to whether it was worth keeping a stock of those steels which were only infrequently called for by customers. An alternative would be to place a demand for the steel on receipt of a customer's order; it could then be processed 3 weeks later.

4.1.4. The reduction in maximum stock from 9,000 tons to 3,100 tons represented a maximum extra availability of working capital of about £180,000; the average amount of capital freed would be about £100,000.

4.2. *General Conclusions*

4.2.1. When the rate of delivery of raw material varies due to factors outside the control of the works, and the rate of consumption varies, a suitable method of stock control cannot be worked out merely from average rates of delivery and demand. Account must be taken of the variations.

4.2.2. Methods of mathematical analysis can be used to deal with some of the situations of this sort. However, these techniques could not be used in the study described in this paper without making unacceptable simplifying assumptions.

4.2.3. An alternative approach is the method of simulation which has been described.

The Evaluation of Possible Alternative Forging Techniques[1]

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This paper describes the early stages of a long-term operational research project which is concerned with increasing the efficiency of the forging industry. The aims of the work have been to provide a yardstick by which individual forges can judge their own efficiency; to provide information which will indicate which lines to follow to effect improvements; to assess the likely effects of current research; and to evaluate possible future lines of research.

The first two aims have been satisfied by comparative studies, and the second two by a process of "breakdown and synthesis". A characteristic of operational research in a field such as this, is the need to acquire an understanding of the technology of the processes involved.

1. Introduction

1.1 This paper is intended to describe the early stages of a particular operational research project; it is in no sense a case study since, at present, we have done little more than set down the way we propose to tackle the job. It may, however, be of interest because it indicates some of the difficulties which are likely to be met in practice, and some of the compromises that have to be made.

1.2 Forging is an unpopular job in the steel industry because it is carried out in relatively uncomfortable circumstances. It is therefore becoming increasingly difficult to find men who are prepared to take it on. The BISRA Operational Research Department have been asked to study the effects of mechanizing the process with a view to reducing the need to employ manual labour, particularly skilled labour. Mechanization may not be the only way to achieve this end, and we can therefore consider the effects of different forms of control, organization and layout.

1.3 Companies will not make changes regardless of cost, so that cost must be one of the criteria by which an alternative technique is judged. The need for costing creates a dilemma. On one hand, since the project is intended to be industry-wide, every firm will stand to benefit from any

[1] Paper presented to the Operational Research Society.

[2] Now with ORbit (Operational Research) Ltd.

[3] Now with Ford Motor Co. Ltd.

favourable results, and therefore has an incentive to supply us with data. On the other hand, the companies are in competition, and for this reason they are sensitive about giving information which might incidentally be used for comparing the costs of different firms making the same products. Initially the costing side was therefore played down, but a satisfactory way has now been found for getting over this difficulty.

1.4 Different companies already employ different degrees of mechanization, and the first approach would be to find the effect of increasing mechanization on the criteria laid down. Next, a more detailed study would be made to try and establish reasons for any differences. The results could at this stage be passed to those responsible for research into the development of new techniques. By a process of breakdown and synthesis it should at the same time be possible to forecast the probable effects of new methods.

2. The Forging Process

2.1 The first step in the study involves finding a common basis for comparing different techniques in different forges. To do this it is necessary to have some knowledge of the forging process so that its essential structure can be recognized.

2.2 Essentially forging consists of two processes in sequence: heating the ingot to a temperature sufficient to permit the material to undergo plastic flow when deforming force is applied, and then hitting or squeezing it by successive strokes to form the required shape. Inside the furnace the steel is being heated at a rate such that the stresses induced by the temperature gradient do not exceed a permissible maximum. While being transported from the furnace to the hammer or press, and while being forged, it is cooling, until eventually its temperature state is such that it can no longer be forged without risk of damaging the material. The quantity of heat which has to be put into the ingot depends on the size of the ingot and the time taken to transport and forge it, since this time determines the amount of heat lost. In practice, however, there is a temperature above which the material to be forged should not be heated.

2.2 Forging has two aims. At the same time as the material is being worked to the desired shape, forging alters its internal structure as cast, and gives it various desirable physical characteristics. Little seems to be known of the amount of working which is necessary to achieve these physical properties, and it appears that the second aim is satisfied almost

incidentally. The amount of working that the material receives obviously depends on the difference in the shapes of the ingot and the end product, and hence the initial selection of the ingot has an important bearing on the achievement of this aim.

2.3 At the end of the forging process the job is generally cut to length and any faults corrected while it is still hot.

2.4 The forging process itself is generally controlled by hand, and the end product is thus relatively inaccurate. It is therefore usually machined to bring it within the specified tolerances. The degree of accuracy to which one should forge should obviously depend upon the relative costs of forging and machining.

3. The Basis for Comparison

3.1 To establish a basis for comparing different techniques, the various processes geared to the making of the desired end product must be considered; for this paper it is convenient to work back from the forging itself.

3.2 The amount of work required to form a given shape depends upon

(a) the size of the workpiece

(b) the difference in shapes between the ingot and the end product

(c) the characteristics of the shape itself.

3.3 The time needed to complete the forging will depend on the capacity of the labour and equipment, and on the maximum permissible working rate of the material, in addition to the shape and size characteristics.

3.4 The amount of heat which must be put into the ingot depends on the forging time (hence on the factors mentioned above) and transporting time, since during these times the ingot is losing heat to the atmosphere at a rate which is probably approximately a negative exponential in accordance with Newton's law of cooling.

3.5 The time taken to impart this heat to the ingot depends on the maximum permissible heating rate of the steel, and the ability of the furnace to heat at this rate.

3.6 These amounts of work and heat which have to be "put into" the ingot, and the times taken to do this, determine the cost and rate of producing the forgings. The amounts will depend on the nature of the work, and the times on the capacity of the men and equipment. The "efficiency" of various combinations of men and equipment could be measured by considering the cost of performing similar amounts of work. This cost will

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be a function of the production time and the cost per unit of production time.

3.7 It would thus be convenient to develop a scale of complexity of different forgings, which would make it possible to compare forges whose products were at different points on this scale.

4. The Complexity of a Forging

4.1 Encouraged by scales of complexity which have been devised for closed die forging, using the constituent geometrical shapes of the forging, we made attempts to develop similar scales for open die forgings. But the decision whether, say, a disc is more complex than a block must, it seems, be based on relative forging times, and forging time is one of the variables we have to study. This circularity led us to abandon this line of approach.

4.2 Failing a scale of complexity defined in quasi-geometrical terms, and in view of the rudimentary state of knowledge of the physics of forging (which might be an alternative route to a scale of complexity), we were thrown back upon the empirical method of analysing the forging process itself, in which the experience of the forgeman is relied upon to a very great extent. The basic data for this will be the actual production times for identical jobs at a number of forges.

5. The Operational Approach

5.1 The selection of forges for more detailed study is governed by one overriding condition: between them they must be representative of the industry as a whole in the sense that they must produce all the basic forging shapes on the market.

5.2 Within these broad confines there are a number of considerations which are helpful in reducing the investigation to manageable proportions.

5.3 In the first place, we need not concern ourselves with every single forging shape, but can select the shapes which occur most frequently in their discrete sector of the range.

5.4 Secondly, those forges are to be primarily considered which are finding it most difficult to obtain labour and are most anxious to reduce costs—if this aim is compatible with the other. In this respect advice of the representatives of the industry has been found indispensable and invaluable.

5.5 Finally, since it is intended ultimately to specify the reasons for differences between techniques, it is appropriate to concentrate on forges which vary most widely in their technical characteristics.

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6. Results so far

6.1 Only a few studies have so far been made—too few to draw general conclusions, but enough to indicate the variables that have to be considered and ways of measuring them, and to show the sort of results that can be obtained.

6.2 The forges studies comprise one medium press shop and four light forges of which one was a press shop and three were hammer shops. The observations made in the heavy press shop were incidental to the work of another department of BISRA, but provided an illustration of the likely effects of changes in the equipment of such a forge. The other four studies were concerned with the forging of similar jobs in four different forges, as part of the programme of inter-firm comparisons.

6.3 A simple method of costing was applied to all the forges studied. This avoided difficulties that may have arisen through using costs obtained directly from the firms whose accounting systems may have differed. The costing used also overcame difficulties connected with publication of results, since it provided a scale for comparison without disclosing the "real" costs at any one forge. It also allowed costs to be allocated to particular jobs, a distinction which is seldom possible with the more usual accounting techniques.

6.4 In outline, the method was as follows. For each piece of equipment, annual costs were calculated under the headings labour, maintenance, fuel and capital cost. These figures were based on information supplied by the firms, with standardization of labour rates, method of depreciation etc. where necessary. Then, from works records and time studies the coefficient of utilization, and hence the number of active minutes per annum was obtained. Dividing the annual cost of a piece of equipment by its number of active minutes gave the cost per active minute for that equipment. The length of time for which each job occupied the equipment was observed during the study. Hence, multiplication of the time of occupation in minutes by the cost per minute gave the cost attributable to each job at each piece of equipment. The cost of each forging was then estimated by adding the costs incurred at each piece of equipment used in its production.

6.5 *Results of a Study in a Medium Press Shop*

6.5.1 Results relating to the medium press shop were treated in two ways to serve two purposes, namely

(a) to show which parts of the forging cycle offered most scope for making savings,

(b) to indicate the effect of making specified changes.

6.5.2 The first type of result is shown in Table 1. This breakdown of costs shows that for the jobs studied in this forge the greatest scope for saving is by improvements in the heating part of the cycle. More detailed analysis showed that a large part of heating costs arose under the heading of fuel, and with other results this led to an increased effort in research into ways of increasing the thermal efficiency of furnaces.

TABLE 1

ALLOCATION OF FORGING COSTS IN A MEDIUM SIZE PRESS SHOP

Job Number	Percentage of Cost attributable to:		
	Heating	Transporting	Forging
1	67	1	32
2	74	3	23
3	62	1	37
4	70	1	29
5	89	Negligible	11
6	56	1	43
7	53	2	45
8	69	1	30
9	82	1	17
Average:	70	1	29

Note. The jobs studied were all different, so that the "average" shown can only be regarded as a rough guide.

6.5.3 The second type of result was derived from detailed observations made at the press by the Plant Engineering Division of BISRA during the forging of a "roll" for a rolling mill. They noted the following points:

(a) The "bite" (i.e. the length of the forging moved into position under the tool between strokes) varied between 2 in. and 10 in., averaging 6 in., whereas the tool was large enough to allow 11 in. bites each time.

(b) Ram-pressure reached only 20 per cent of the designed capacity, so that pressing time was unnecessarily long. (This of course was an engineering failure, not a matter for operational research.)

(c) The tool was being raised higher than was necessary for the movement of the workpiece.

6.5.4 The effect on forging time of the introduction of a mechanism for

controlling the position of the ingot was estimated for three conditions. These conditions were:

(i) manual control of the press (as during the studies)
(ii) operation of the press at its maximum speed
(iii) press operation linked with the ingot position-control mechanism.

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6.5.5 The first condition would allow the maximum bite of 11 in. for each stroke except end strokes, so reducing the number of blows needed to complete the forging from 226 to 135, and saving 8 min, or 20 per cent of the forging time. The second condition would allow this reduced number of strokes to be made with a shorter stroke cycle time, making a total saving of about 25 per cent. The third condition would synchronize the movement of the tool and ingot, so that the total reduction in forging time compared with the existing method would be about 30 per cent.

6.5.6 The value of such a saving was assessed in terms of the heat saved. Heating the ingot through the forging range of temperatures accounted for about $\frac{1}{7}$ of the total heating cost. A 30 per cent saving in forging time would thus save about 30 per cent of $\frac{1}{7}$ of the cost of heating. This saving would probably not justify the expense of making the modifications at the press

6.5.7 The possibility of increased throughput was also considered, but was found to be slight with the existing furnace system. Although a 30 per cent saving in forging time could be made, the corresponding saving in heating time was only 30 per cent of $\frac{1}{7}$ of the total heating time. Therefore increased furnace capacity would be required to take advantage of the increased press availability. In fact, additional furnacing or reduced heating times would have increased throughput far more than speeding up the press, since furnace utilization was almost 100 per cent, while press utilization was below 20 per cent mainly through waiting for ingots to reach forging temperature. Improvements in the forging part of the cycle were therefore unlikely to pay until the heating part of the cycle had been improved.

6.6 *Results of the Studies in Four Light Forges*

6.6.1 To assess the effects of varying degrees of mechanization, different layouts etc. and to investigate the existence of differences of forging technique, a series of studies was planned, during which the production of similar forgings would be observed in different forges. Four of these studies have been made, and some early results are presented in Table 2. The job studied was the production of 4 in. square billet from 9 in. square

ingots of a special tool steel.

6.6.2 The cost of heating an ingot was calculated for each forge by the method described in Section 6.4, and is shown in Table 2. Differences in the costs are due to a combination of two factors

(a) different costs/hour/ingot for running the furnace

(b) different times in furnace.

6.6.3 No explanation is offered for the differences in heating time, which may be due to furnace performance, management policy, production scheduling or some other reason. The different breakdown between preheating and forging cost in forge 4 can be explained, however, by the fact that the cooler end of the so-called "forging" furnace was at a temperature about 200°C lower than the corresponding end in other "forging" furnaces. The figures quoted are averages, and in all forges there were considerable variations from one ingot to another. For example, ingots of the same size and cast were heated in one forging furnace for periods between 2½ and 5 hr.

TABLE 2

HEATING TIMES AND COSTS FOR TOOL STEEL INGOT APPROXIMATELY 9 IN. SQUARE

Type of Furnace	Forge 1		Forge 2		Forge 3		Forge 4	
	Time (hr)	Cost (£)	Time (hr)	Cost (£)	Time (hr)	Cost (£)	Time (hr)	Cost (£)
Preheating	16	—	12	0·57	13	1·26	6	—
Soaking	—	—	5	0·25	—	—	—	—
Forging	4·25	1·0	3·25	0·55	3·0	0·54	20·7	1·05
Total	20·25	1·0	20·25	1·37	16·0	1·8	26·7	1·05

Note. The "zero" cost of preheating in forges 1 and 4 is assumed because preheating consisted of placing the ingots in the path of the furnace exhaust gases.

6.6.4 The furnaces with lowest cost were both continuous furnaces, but this may not be a general rule. In all cases the major cost was fuel, which constituted up to 70 per cent of total furnace costs.

6.6.5 Forging costs were also calculated by the method described in Section 6.4 and the results are shown in Table 3.

6.6.6 The total annual costs of running the actual forging equipment were very similar in all four forges, yet, as shown in Table 3, the cost per minute varied considerably. This was due mainly to different degrees of

utilization, of which the cause could only be revealed by further study. Possible causes are furnace performance, work scheduling, variations in fatigue or morale in the crews, etc.

6.6.7 The differences in forging times are due in part to the suitability of the equipment for the work being done. For example, the equipment in forge 4 was used mainly for work of a larger size, and may have a better performance than the other forges for other types and sizes of work.

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TABLE 3

FORGING TIMES AND COSTS FOR COGGING TOOL STEEL INGOTS TO 4 IN. SQUARE

	Forge 1	Forge 2	Forge 3	Forge 4
Forging time (min)	10·7	9·3	11·0	15·2
Forging cost (£/min)	0·095	0·102	0·132	0·121
Forging cost (£)	0·98	0·96	1·45	1·84

7. Use of the Results

7.1 The results themselves have already been of some use. For example, within BISRA they have encouraged greater research effort into the problems of heating for forging. More generally, they have aroused interest from the industry which should facilitate extension of the work.

7.2 Perhaps more valuable than the results is the experience that has been gained within the Operational Research Department. Some technical background is needed for a project of this type, and this has been acquired during these early studies. Also the important variables have now been identified, and ways of measuring them have been devised. One method that did not pay off was filming a hammer at work using a normal speed cine camera, in the hope of obtaining information about the variations in its operation. The analysis of the film was such a lengthy process that this method was discontinued. On the other hand, the method of costing that was used has aroused considerable interest and may be adopted more widely.

Operations Research for Computer Control

GEORGE H. AMBER* AND PAUL S. AMBER*

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Computer control represents a high order of automatism. But before a computer can be used to control an industrial machine or process, the mode of operation of the machine or process must first be expressed as a mathematical equation. Control engineers are familiar with computers and servomechanisms, but are not especially able at writing the process descriptive equations on which computer control is based. Writing the mathematical models of a machine or process, thereby making the machine or process amenable to automatic computer control, is a job for O.R., operations research.

THE concept of basing control of an industrial process or machine (hereinafter process includes machines too) on the automatic solution of process descriptive equations, to attain a high order of automatism, has existed for some time now. But, before process equations can be solved, they must first be determined. This obvious fact is the soft spot in computer control for automation: Who can set up the necessary equations?

The engineering arts of putting equipment together to solve most any equation is already highly developed. But the control engineer does not necessarily have the skills of analysing a process and reducing it into the form of a mathematical model. This is why computer control, a powerful automation technique long recognized, has developed so slowly.

On the other hand, some operations research aspirants claim that there are insufficient demands for their operations research skills. So we have the situation that automation computer control suffers from lack of mathematically oriented process analysts, while some operations research workers can't find enough to do along such lines. What the automation computer-control designer needs is the aid of the professional equation writer. This is the operations research mathematician.

This isn't to say that the engineer is necessarily the potential client of the mathematician. Rather, the automation design engineers and the applied mathematicians should join forces, under the broad discipline of operations research, and jointly work to advance computer control automation for the processing and fabrication industries.

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The Route to Computer Control

Briefly, here is the line of reasoning we, the authors, followed, that led to our convictions for computer control. We observed that all work consists of but two basic elements: the energy that does the actual work, and the information on how this work is done. For all practical purposes, energy is about fully mechanized, for there is little use of muscle power nowadays. But we are just now entering into the era of mechanization of information, which is accomplished by automatic control systems, including computer elements.

However, mere mechanization of obvious routines (such as statistical quality control) does not seem to be a sufficient effort. Rather than such strictly passive measures, the active application of the "scientific method", for use in the production plant, is advocated here.

The "scientific method" (oversimplified, no doubt) consists of providing general solutions (usually in the form of equations) to a "process", by proper isolation, evaluation, and analysis of the process constants, parameters and variables. This approach, writing equations of a seemingly unpredictable process, generalizes the process. This is equivalent to routinizing the process, because the equation can be solved in a routine manner, after the values of the factors are merely plugged-in". Thus, a varying process can be controlled automatically, by means of a *mentor* (control computer) that continually reconciles the process factors; constants, parameters, and variables. Not infrequently, the solution of a battery of simultaneous process equations is involved in the computer-control scheme.

The only limit to the mechanization of information, by means of control computers, is the mathematics involved. What physical or industrial process, including mental operations, cannot be expressed in mathematical form? *The only real problem is locating the problem*; finding a potential area for computer control. Then the applied mathematician will seldom be unable to express the process in mathematical form. Following this, the system control engineers can shape up the computer hardware, largely from off-the-shelf components.

Computer Control

Because of the interest of the press in "Giant Brains", computer control may bring forth visions of large differential analysers or digital computers.

True, such large *general-purpose* analog and digital computers are useful for control purposes. However, it is *limited-purpose* computers that are more likely to be used for control purposes. Furthermore, *equation solvers*, rather than *computers*, are used for control. Such equation solvers (as compared to differential analyser computers) are usually "special-purpose, electro-mechanical, servotype, analog, equation solvers". This is why the authors prefer the short term *mentors*. Yes, limited-purpose digital computers too, are used for control purposes.

As system engineering consultants, our big problem so far has not been the engineering involved in designing a computer control system, but abstracting the mathematical formulae that define the operation of the process to be controlled.

We could see, as already pointed out, that the practical approach to computer control automation is that the process control engineers and mathematicians both be members of the control system design team, and meet on the common ground of operations research. In fact, we view computer control as a specialty falling into the broader discipline of operations research.

The authors appreciate the capabilities of operations research, but believe that it is not living up to its potentials. One reason is that operations research seems to be considered suitable mainly for advising high-level executive decisions. A practically unlimited area for operations research consists of writing equations of everyday industrial processes, to make them amenable to computer control. Needless to say, not until machine builders have been educated to design their machines and manufacturing processes on an operations research basis, will computer control be commonplace.

Letters to the Editor

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Operations Research for Computer Control

IT seems to me that G. H. and P. S. AMBER are stating the obvious in their article on computer control. I cannot imagine how any process can be successfully automatized without prior operational research for the purpose of obtaining a logical (i.e. mathematical) description of it and indicating an optimal solution. It is certainly true that, in the one field in which it is already usual to use computer elements for control purposes, namely guided missiles, operational research sections form part of the design teams.

What interests me is, seeing that the authors are American—and we tend to feel that Americans are so very much better than we are in adopting new methods—why did they feel the need to write this? Is it possible that they were prompted by the recognition that professional jealously between system engineers and operational research people is prevalent in the U.S.A. and ought to be stopped?

E. KAY

Imperiai Court,
Kennington Lane,
London, S.E.11.

MR. KAY states that he cannot imagine how any process can be successfully automatized without prior operations research. We quite agree with him that big feed-back control systems, and computer-controlled systems, should be based on the sanctions of operations research. Why else did we write our paper? But those homey automation systems that manufacture roller-skates, shoe polish, locks, valves, and peanut butter, do not use operations research, as you can well imagine. I am very well informed on "Detroit" automation, and deplore the fact that it is almost entirely based on empirical cut-and-try design methods. Mr. Kay further suspects a rivalry between systems engineers and operations research people. Such "jealousy" would at least imply that they are aware of each other's existence and activities. Alas, some operations research men (like

Mr. Kay?) have no interest in the conditions that prevail on the factory floor. And, most engineers seem to feel that operations research belongs on the university campus, not on the front lines of industry. I hope my article shows that I encourage operations research men and systems design engineers to work together. Is this not a worthy purpose? By the way, I much prefer a healthy controversy to blank indifference.

We engineers look to our professors and mathematicians (as are found in operations research) for guidance. I hope they do condescend to come down to our level at times, for our mutual advantage. Thank you for hearing-out my candid remarks.

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Abstracts and Reviews

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Sampled-data Control Systems.
ELIAHU I. JURY.
John Wiley and Sons, New York; Chapman and Hall, London, 1959. xi + 447 *pp.* 128*s.*

The work is a treatment of material contained in various papers, including recent research carried out under the direction of the author, and arranged into a graduate lecture series.

The theory on sampled-data systems has been evaluated and presented as a unified treatment. It is directed to graduate students and is helpful to engineers and scientists applying themselves to sampled-data systems.

The work is largely theoretical and lays the foundation necessary for a thorough understanding of sampled-data systems. The treatment of the z-transform and the root locus plot in the z-plane are exceptionally well presented and illustrated. Discrete and continuous compensation procedures and finite pulse width systems are treated rigorously and exhaustively.

The basic theory developed may be applied not only to sampled-data systems but also to allied fields such as computers. It is somewhat regrettable that sufficient practical systems are not investigated in detail to make it more attractive and useful to the engineer rather than directing the work as a whole to the needs of the instructor and degree student. Nevertheless, the ideas presented form the basis for not only pursuing further studies and research but also applications in industry in the solution of problems.

W. L. M.

Principles and Applications of Random Noise Theory.
JULIUS S. BENDAT.
John Wiley and Sons, New York; Chapman and Hall, London, 1958. 431 *pp.* 88*s.*

This book was written primarily for engineers and opens with three chapters introducing the concepts of stationary random processes, auto and cross-correlation and power spectra. They also include a summary

of the probability theory needed in random noise analysis. The treatment does not demand a wide mathematical background and is designed to be illuminating rather than rigorous. Chapter 5 presents empirical and theoretical arguments for the wide applicability of an exponential–cosine auto-correlation function. Chapter 10 deals with the distribution of zeros of a fixed sine wave or of a sine wave process accompanied by random noise. The rest of the book is concerned with applications of the theory to electrical engineering problems. Chapters 4 and 9 deal with the characteristics of optimum filtering systems and extend the work of Wiener in this field. Chapter 6 considers analogue methods of examining the effect of a given filtering system. Chapters 7 and 8 discuss the measurement of auto and cross-correlation and the effect of such factors as correlator bandwidth and smoothing time.

Much of the subject matter of later chapters is relatively new. It demands a knowledge of Laplace transforms and of circuit theory and is likely to be of interest mainly to the specialist. The introductory part, being concerned with ideas of greater generality, will be of more interest to operational research workers. This work is well presented, and the author's claim that no previous knowledge of probability theory or Fourier integrals is needed is justified. The book should thus prove a useful introduction to the subject of random noise for a wide variety of readers.

J. H. C.

Sampled-data Control Systems.

J. R. RAGAZZINI and G. FRANKLIN.

McGraw Hill, London, 1959. 331 *pp.* 74*s.*

The technique of sampled-data control system design is presented in a manner which is clear and precise throughout. A great deal of development work on this subject carried out under the direction of the senior author has appeared recently in *Transactions of the American Institute of Electrical Engineers.*

In spite of the fact that the treatment is mainly theoretical, the subject is broadly treated in a systematic and meaningful manner. Such aspects as the interpretation of sampling, data reconstruction, and digital compensation of sampled-data systems with which most designers are to concern themselves, are developed logically so as to lead the reader into a thorough understanding of the principles involved. The more general

aspects of sampled-data systems such as behaviour between sampling instants, multi-rate sampled systems, and systems with random inputs, complete the subject and bring the text into line with the latest work.

The z-transform is used throughout, and the various theorems upon which performance is based and design carried out are lucidly explained. A feature of the book particularly valuable to beginners in sampled-data control systems is the way the author relates the subject of the z-transform to the Laplace transform approach, now familiar to all workers in the analogue control field. Besides providing an introduction for the engineer, the work is of a sufficiently advanced nature to be useful as a reference on the subject for the research worker.

The design procedures developed theoretically are illustrated by carefully chosen worked out examples throughout the book. These form an invaluable adjunct in emphasizing principles and adding interest to a most useful publication.

W. L. M.
L. N. B.

Operations Research for Industrial Management.
DIMITRIS N. CHORAFAS.
Rheinhold Publishing Corporation, New York, 1958. *Chapman & Hall, London*, 1959. vi + 303 *pp.* 70*s.*

The first chapter of this book explains very clearly and concisely for non-scientific consumption the purpose and what might be called the philosophy of operations research; it is a first class essay. Unfortunately the rest of the book does not live up to it.

The three chapters which follow are so full of misprints that some of the mathematics becomes nonsensical; the proof reading throughout the book is well below what one has a right to expect.

Much is made of "simulation" throughout the book which, considering its title, an operational research analyst would not find surprising except that the term is used as a general one covering the mathematical description of a system; this may be purely a language difficulty, but there is hardly a mention of a "Monte Carlo" in the book.

The third chapter, entitled "Simulation and its pitfalls" is wholly concerned with relating two case studies illustrating how not to do operational research. However, the reader will look in vain for a complementary

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chapter showing how these cases should have been tackled. Any merit the cautionary admonitions might have had is thus entirely lost.

The book also contains chapters on the theory of games, matrix analysis, linear programming, the flow technique and several on business gaming. Some of this is interesting and well supported by worked examples although at least one of those on games theory is inapplicable and results in a manifestly false answer.

However, there is hardly any mention of probability theory and none of queueing theory and management, one feels, would obtain a very biased view of operational research. I cannot recommend this book to anyone, least of all to industrial management.

R. H. COLLCUTT

The Advanced Theory of Statistics. Vol. 1: Distribution Theory.
MAURICE G. KENDALL and ALAN STUART.
Charles Griffin & Co. Ltd., London, 1958. xii + 433 *pp.* 84*s.*

Many readers of this journal probably use the tools of mathematical statistics in the course of their work, and in so doing will be familiar with Vols. 1 and 2 of "The Advanced Theory of Statistics", which first appeared in 1943 and 1946 respectively. These contained a broadly based account of theoretical foundations, with many applications which are typical of the day-to-day work of the research worker. The new Vol. 1, in which Professor M. G. Kendall has been assisted by Mr. Alan Stuart, is the first of three which are intended to be a complete revision, taking account of the advances and expansion of the theory and use of statistical methods. It covers the theory of distributions, including sampling distributions. The second volume will deal with estimation, inference and statistical relationship. Vol. 3 will deal with the theory of the design and analysis of sample surveys and experiments, multivariate analysis and time series. We are warned, however, that it may be several years before the complete work appears, and Kendall hopes that the original Vol. 2 will be useful in the meantime.

The book is essentially a mathematical treatment, with considerable emphasis at times on full mathematical rigour. To obtain the maximum possible benefit from the book, the reader should be familiar with multiple integrals, Jacobians, matrix algebra, contour integration, convergence theory, etc. In other words, the standard is that of an Honours Mathe-

matics degree, although the scope is probably wider than that included in any degree course in statistics. This is not to say that the reader with less equipment will not find the book useful. The foundations are clearly and simply discussed, there is a wealth of examples and exercises, and a fairly complete set of working tables. It should also stimulate interest in theory among those who, at the moment, are only interested in practical applications.

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The main theme of the book is concerned with populations, their specification and properties and those of the derived sampling distributions. Chapter 8 is a brief discussion of the reverse class of problem, that of making inferences about a parent population, not completely specified, on the basis of observations which emanated from it. A systematic study of this will be begun in the second volume and it only appears in broad terms here to give point to what is discussed later in Vol. 1. Chapter 7 discusses the calculus of probabilities in such a way that inference is later treated systematically. Some writers use the "theory of probability" to cover all uncertain inference, building up the theory with axioms and taking "probability" to be an undefined idea. The second approach defines probability in terms of the relative frequency of events and refers the theory to the pure mathematics of set or sequence theory. The authors' view is that neither approach is sufficient by itself to the statistician:

"It seems, however, that every man must choose for himself and that his psychological make-up, his experience and his fields of interest all determine the kind of axiomatization which he prefers. In statistics it is a mark of immaturity to argue overmuch about the fundamentals of probability theory."

It seems that when any pair of mathematical statisticians cannot agree on how to interpret some body of data they should examine the fundamentals of their theories and if necessary agree to differ at that level. Human preferences must be made clear at this level, but the authors point out that, at a different level, that of sampling operations, human preferences must be kept under control and bias must be avoided (para. 9.6).

There are one or two points on which the reviewer would like to have seen further discussion. It is perhaps inevitable in mathematical statistics that the choice of a particular expression to represent a frequency distribution is often based on partly arbitrary grounds. Where, therefore, there are theoretical reasons justifying the use of a particular distribution, as

is stated on p. 129 in respect of the Type III distribution of accident proneness in a human population, it would be interesting to know what these were. Similarly, it would be useful to see some discussion of the relation between the distribution of strengths of materials, the "weakest link" theory and the distribution of extreme sample values in Chapter 14.

All operational research workers will be indebted to Kendall and Stuart for their painstaking work and will await with interest the later volumes.

F. GARWOOD

Elementary Mathematical Programming.
R. W. METZGER.
John Wiley & Sons, New York, 1958; *Chapman & Hall, London*, 1959. vii + 239 *pp.* 48*s.*

The clouds of mystique which surrounded linear programming in the earlier days have largely dispersed and the publication under review is a symptom of the present state of evolution. It is "designed for the reader with a limited background in mathematics who wishes to understand the basic techniques and applications of mathematical programming". It is unfortunate that a popular exposition should be handicapped by an inaccurate title; the book deals exclusively with linear programming and makes no pretence to cover other forms of mathematical programming.

The treatment, which lays stress on methodology rather than theory, starts as usual with the transportation problem and leads on to the general linear programming problem and its solution by the simplex method. The later chapters deal with the practical solution of a number of problems. A rather unusual aspect of the book is the inclusion of sections dealing with approximate methods of solution. These should be of interest to readers already having some knowledge of linear programming. A difficulty with approximate methods is that they tend to be specific to particular problems and to fail on more general application, but it is encouraging to find the theoretical approach being used alongside "quick and dirty" methods.

The examples succeed in giving a confident tone to the book and should leave the reader for whom it is designed with the feeling that linear programming is a useful tool and not simply a complicated mathematical exercise. If it finds its way on to an operational research book-

shelf, it is likely to be consulted surreptitiously by the more advanced reader as well.

J. STRINGER

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Operational Research (Betriebliche Verfahrensuntersuchungen).
E. LIEBEL.
Ergmetall, **11**, 438–446 (September, 1958). (In German.)
The development of operational research from the war years to recent industrial applications is outlined, and two examples connected with the mining industry are considered. The first of these sets the problems connected with the transport of ore from five hypothetical ore preparation plants in different parts of the world to an iron works in Germany, while the second works out in detail a complex problem concerned with the transport of stone from a works to three plants for treatment, and its subsequent dispatch to five receiving areas. The method of applying operational research techniques to the problem, including the use of calculating machines is given, together with the costs of the method (29 references).

Queueing Theory and Water Storage.
W. B. LANGBEIN.
Proc. Amer. Soc. Civ. Engrs, **84**, HY5 (*J. Hydraulics Division*), Paper 1811 (October 1958).
The application of the "queue theory" approach to problems of water storage in hydroelectric, flood control, and irrigation systems leads to a general method of calculation called "probability routing". The paper might more appropriately have been entitled "A Queue Analogue to Water Storage", for the development does not follow the usual developments. An understanding of queue behaviour is essential background for this reasonable and practical approach to water planning. The queue analogy is expressed in the following terms: (a) queue arrival rate = inflow; (b) service function = discharge; (c) queue discipline = order of drawing discharge (selecting drafts may be made from high or low levels to draw off warm or cold water, turbid or clear water, or water differing in salinity); and (d) attrition rate or evaporation rate. Analytic and computing methods are discussed and several complete examples are used as illustrations.

The probability routing method is a sort of single-step queue equation solved by the method of finite differences. The data required are the probability distribution of inflow and the discharge function (which is generally a function of the storage). One of the examples is based on a 54-year record of inflows which allows a reasonably reliable distribution function for annual flows. The probability routing method does not suffer from a number of inadequacies of previously used methods (for example, it has been shown that the storage needed, as calculated by "trial routing", increases with the 0·72 power of the length of the historical record). The paper should be of great value in the future and represents a new development in queue theory.

On the Economic Management of Large Organizations: A Laboratory Study.
S. ENKE.
J. Business, **31**, 280 (1958).

A descriptive account of the study made by the RAND Corporation of the most economical method of controlling spares in the U.S. Air Force. The basic considerations and the types of data required for solution are clearly set out, but little is said of any mathematical methods involved. In the light of the study of the situation in the Air Force, it was concluded that most of the re-ordering should be done automatically by means of data-processing centres, on criteria derived from past experience of failure rate and on cost assessment of the consequences of different degrees of shortage of spares, and with procurement of the main supplies of high-value spares deferred till some operational experience had been gained. When the proposals had been formulated they were tested in an experiment that combined aspects of a game and a simulation, in that two teams simulated the procurement decisions of similar squadrons under similar conditions of flying hours, breakdowns, and so on, and with similar random "interferences" (delays in supply, etc., and eventually two wars). The exercise both showed a marked saving in cost of procurement for the new method, with little fall in effectiveness of the unit, and also showed a clear tendency of the teams to learn to administer either system as they went along. In the case of the new method, a part of this learning was, in fact, the discovery of how to put more of the work on to the computer.

News and Notes

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Regional Research Laboratory, Hyderabad

Annual Report 1957–58

THE report states that the methods of operational research have been used since 1949 in this Laboratory in an attempt to consider the possibilities of an industrial problem, even before it is taken up, from an integrated point of view: this includes past work on the problem, plan of research, its place in the pattern of work of the laboratory, eventual production possibilities on a pilot-plant and by industry, likely hitches in the way of raw-material availability and costs, social factors and other questions. The operational research team comprises the actual research workers, chemical engineers, operational research specialists and the Director; any problem that comes up is carefully considered on the basis of a literature note put up by the worker and if the work is undertaken, parallel schemes comprising laboratory research on the problem, obtaining of pilot-plant specifications, raw material costs and market data, surveys by questionnaires or by visits to industry, geological or mineral survey data, etc., start functioning simultaneously. Some of the pitfalls in the utilization of industrial research are thus circumvented at an early stage and a clear overall picture of the work in its industrial setting in the country emerges.

On the other hand, once work is under way the operational research unit in periodic meetings evaluate the progress of work, prevent unprofitable side-shooting and integrate within the laboratory the work on problems coming under the purview of various disciplines.

When the laboratory work is complete and after pilot-plant work has proved the economic and technical feasibility of a process, a non-technical note is drawn up presenting the process for replacement with industry. At another level, the laboratory seeks to place its experience and laboratory facilities at the service of the country by advising government, through project reports and project cost studies, of the establishment of new industry.

An important part of the work of the operational research unit is to

study and bring about internal co-ordination between research administrative and auxiliary services in the laboratory. This enables an integrated approach in smooth and speedy progress of research.

The operational research unit also studies the effect of various policy decisions on the progress and direction of research and assists the executive in arriving at these decisions.

Operational Research at Work

A One-day Conference to be held on 17 September
at the Connaught Rooms, London

THE British Institute of Management in collaboration with the Operational Research Society will present a One-day Conference on Thursday, 17 September, 1959.

This Conference, which has been arranged for both management and operational research workers, will concentrate on the application of operational research in industry, with particular reference to stock control, long-range policy planning and transport. Speakers will include Miss Mary Munn, Mr. R. H. Collcutt, Dr. K. Pennycuick and Mr. B. H. P. Rivett. Full details of this Conference and application forms may be obtained from Mr. A. Green, Head of Production Section, British Institute of Management, Management House, 80 Fetter Lane, London, E.C.4.

Mr. Stafford Beer

OUR congratulations to Mr. Stafford Beer for the award of the Silver Medal of the Royal Swedish Academy of Engineering Sciences for appreciation and admiration not only for the lectures, but for the far-sighted and practically important work he is doing in the application of cybernetics and operational research.

These lectures are to be published in book form under the title "Cybernetics and Management" by the English Universities Press.

News of Members

New full members:

Professor M. H. Belz—University of Melbourne.
B. Bishop—J. Lyons & Co., Ltd.
D. A. Bullock—(no business address).
H. C. Calpine, Director of O.R. Admiralty.
M. L. Chambers—Air Ministry.
B. J. Elson—British Petroleum Co., Ltd.
J. A. van der Heiden—N. V. De Bataafsche Petroleum Maatschappij.
J. S. Ogilvie—Hepworth & Grandage Ltd.
R. J. Taylor—B.I.S.R.A.
Dr. Vajda—Admiralty Research Laboratory.
C. A. Warner—U.S. Operations Research Groups.
Dr. W. Young—National Coal Board.

The following movements of members have come to our attention:

J. Banbury on 1 June joins ORbit (leaving B.I.S.R.A.).

B. Bishop, Project Manager J. Lyons & Co., Ltd. (previously with British Railways).

Max Davies is in Nigeria, Public Relations Manager at Owerri with Shell–B.P. Petroleum Development Company of Nigeria Ltd.

H. Dunn is joining Canadian National Railways in Montreal, leaving the British Columbia Research Council.

R. S. Gander, Littlewoods Mail Order Stores (previously with United Steels).

A. Glaskin, Petfoods Ltd. (previously with Vickers Groups Research).

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A Unified Theory for Stock, Storage and Queue Control*

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The basic concepts are formulated of mathematical theory for the study of the operation of a wide class of man–machine systems. Queueing, stock and storage models are special cases of the general probability model which is described.

1. Introduction

IT is my object to present some basic connexions between queueing, stock, storage and allied problems. These connexions are at the mathematical level, but in this paper only a few basic concepts will be explored, the mathematical development of these concepts being left for future papers.

When one considers the various applications I have indicated, one finds that to some extent both methods of approach to problems and also the objectives sought differ in each. By attempting some unification one would hope to extend and enrich the underlying mathematical theory. Now, what is it that these applications have in common which could make a unified general theory possible? It is not just that they are all examples of stochastic processes: it seems to me that the possibility of a general theory with significant content resides in the common feature they all possess of containing a self-regulatory, or feed-back, mechanism. That is, they are all stochastic control processes, usually of discrete type.

Certain *ad hoc* connexions have already been pointed out.[1, 2, 3] However, I believe that these do not go far enough. In my contribution to the discussion on Gani's paper[1] on storage systems, I suggested then that there were further interconnexions still to be explored, but the unifying principle

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eluded me until I heard P. D. Finch read his paper.[4] In this paper Finch considered a certain type of single-server queueing system with finite capacity N and proved the theorem that the distribution of queue-size q in this system is identical with the distribution of $N-q'$, where q' is the queue-size in the dual system which has the inter-arrival time and service time interchanged.

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Finch's duality theorem leads directly to the developments in this paper. With the introduction of some further general concepts, to be explained below, it turns out that the duality result is self-evident, and it thus becomes a principle rather than a theorem.

2. *The Model: Single-channel Systems*

In order to introduce the general concepts mentioned above, I shall apply them first to a model in its simplest, most special form.

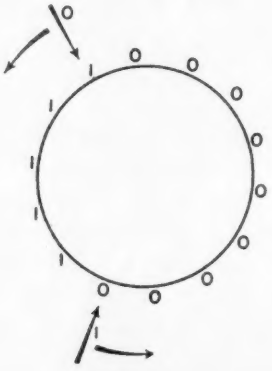

FIGURE 1.

If we think of a simple queueing system, of what does it consist essentially? We could describe it as consisting of a number of positions which may or may not be occupied by units of some kind; that is, we consider a number of elements each capable of registering two states—let us call them "binary counters". We then have a register consisting of an assembly of N such binary counters. We may label the two possible states of each counter 0 and 1 respectively. This is the system, and in the simplest cases, its state is given by the number r of counters currently registering the 1 state; or equally well, by the number $N-r$ registering the 0 state.

We can conveniently picture this model as in Figure 1. Now how does it operate? We suppose that it is acted upon by two types of input, which

we call the 0-input and the 1-input. The inputs come into operation according to certain rules to be described below. The effect of the 0-input is to switch a group of one or more counters, currently registering 1, to the 0 state, and *mutatis mutandis* for the 1-input.

In the operation of the inputs, we distinguish between the rule which determines when an input comes into operation and the rule which determines how many of the counters are switched as a result of the input coming into operation.

We shall define an input as "triggered" if its operation is initiated when and only when the number of 0's and 1's satisfy certain prescribed conditions. These conditions will depend upon the type of application and examples will be given below. If the initiation of the input is independent of the state of the system we shall say that it is "untriggered".

We define the "input quantity" as the number of the counters which are switched as a result of operation of the input. We shall say that the input quantity is "controlled" if the number of counters switched depends on the state of the system, either at the instant of initiation of the input or at the instant of its completion. Otherwise, if the number of counters switched is independent of the state of the system, we shall say that the input quantity is "uncontrolled". A particular case of an uncontrolled input quantity would be a constant input quantity; more generally, it could be a random variable.

The time which elapses from the instant the 1-input is initiated until the counters are switched will be called the "1-input time". The "0-input time" is defined similarly. Once an input is initiated it cannot be re-initiated until the termination of that particular input time: during this time the input is "engaged".

We now consider some examples.

(1) To take perhaps the simplest case, suppose the 1-(0-) input is triggered by the presence of one or more counters in the 0(1) state. Let the input quantities in both cases be constantly unity. With the adoption of a cyclical ordering of the counters, the first counter in the 0(1) state is held locked when the 1-(0-) input is triggered and then switched to 1(0) at the termination of the 1-(0-) input time. Let the 1-(0-) input time be a random variable, $\tau_1(\tau_0)$. This then is a model for a type of finite queue. We can suppose that the 1-input operates the arrivals and the 0-input operates the departures. τ_1 is the inter-arrival time and τ_0 is the service time. The presence of a unit in the system is indicated by a counter registering 1, and a vacant place is indicated by a counter registering 0. The service mechanism is

initiated, when it is disengaged, as soon as there is one unit in the system: if there are no units in the system the service mechanism stands idle. Similarly, the arrival mechanism, when it is disengaged, comes into operation as soon as there is one vacant place in the system: if there are no vacant places, i.e. if there are N units in the system, the arrival mechanism is frozen until there has been a departure from the system.

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There are, of course, various other possibilities for the input rules in accordance with which the 1-input and 0-input could operate. Let us consider some of these.

(2) Let the 1-input be untriggered. That is, it just occurs from time to time with an interval according to τ_1. Let the 1-input quantity be controlled according to the rule that if there are some counters in the 0 state at the termination of an input time, the first one is switched, otherwise nothing happens. If the 0-input is triggered as before, this model would correspond to a type of finite queue in which, when the system is full, an input is virtual, but the event can occur, i.e. there are "lost inputs".

A very special case of this model is where $N = 1$. This is the case of the so-called Type 1 counter problem, which has been studied in connexion with Geiger–Müller counters, usually under the added assumptions that $\tau_0 =$ constant and τ_1 has an exponential distribution. Radioactive disintegrations provide the input. An input is registered only if the counter is free (in the 0 state). It locks the counter for a constant time, when it then becomes free again.

(3) If the 0-input is also untriggered, with controlled input quantity as above, we have a model for a storage system. Units arrive, or virtually arrive, with a certain inter-arrival time distribution, and are withdrawn, or virtually withdrawn, with a certain inter-departure time distribution.

(4) More generally, suppose that while the 1-input is as above, and the 0-input is untriggered, the 0-input quantity is controlled to be equal to the total number of counters in the 1 state at the termination of the 0-input time. If we specialize τ_0 to be constant, and τ_1 to have an exponential distribution, this would be a model for the so-called cyclical review system of stock control, in the case where the lead time is zero. The 0's represent the items in stock. Demands for items occur at random, and at constant intervals of time the stock level is inspected and immediately brought up to a fixed level N, by replacement of the items withdrawn since the last inspection.

(5) With the 1-input again as above, suppose that the 0-input is triggered by the presence of exactly k counters in the 1 state, and let the 0-input

quantity be controlled to be constantly k. Thus at the beginning of a 0-input time, k 1's are locked and switched to 0's on the termination of the input time. This would be a model for a queueing system with batch-wise service. The service mechanism does not operate until at least k units are waiting, and they are then all served simultaneously.

(6) With the 1-input as above, let the 0-input be triggered by the presence of one or more counters in the 1 state, and let the 0-input quantity be controlled to be k, if there are at least k counters in the 1 state at the beginning of the 0-input time or otherwise equal to the number of counters in the 1 state in the case that there are fewer than k. Thus at the beginning of a 0-input time a number of 1's, between one and k, are locked and switched to 0's at the termination of the input time. This is a queueing model in which service is batch-wise with up to k units being served simultaneously.

These examples should be sufficient to illustrate the possibilities in regard to input rules.

3. *Multi-channel Systems*

The above examples all relate to single-channel systems. Let us now further generalize the model by supposing that there are several inputs of either kind, not just one of each. Such systems will be called multi-channel systems. In considering the input rules we have now to distinguish carefully between those counters in the system which are currently "locked", i.e. those upon which engaged inputs are operating, and the unlocked ones. We consider some examples.

(1) Suppose there is a single untriggered 1-input as in the previous examples, but that there are N 0-inputs, each triggered by the presence of one or more unlocked 1's and with uncontrolled input quantity constantly unity. All 0-input times are random variables, τ_0, independently and identically distributed. This would be a model for a queueing system with N servers and no waiting, of the type studied, e.g. in telephone traffic theory. The 1's represent the units in the system, and since the number of servers equals the number of places for units, there is no possibility of waiting. It could equally represent a stock control system in which an item is re-ordered as soon as it is sold. Here a 0 would represent an item in stock, and a 1 a vacant place for one item. The maximum stock is N, and the lead time is τ_0.

(2) As an example of a more complicated mechanism, suppose that in (1) the N triggered 0-inputs are restricted to commencing operation at

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discrete, equally spaced, time marks. Suppose further that the input quantity is controlled to equal the number of unlocked 1's present at the instant of commencement of the input time. This would be a model for the usual cyclical review system of stock control, with distributed lead time.

(3) As a generalization of (1) suppose that there are s 0-inputs, each triggered by the presence of k unlocked 1's, and that the input quantity is uncontrolled and constantly k. If $sk = N$, this would be a model for a queueing system with no waiting, with s servers and batch-wise service of k units per server. The service mechanism is seen to be triggered only when multiples of k units are present. It is also a model for the so-called re-order level system of stock control. Here again the 0's represent the items in stock. A replacement of k items is ordered whenever the deficiency from N becomes a multiple of k. Alternatively, we can state the re-order rule as: re-order a quantity k whenever the stock on hand plus that on order falls by an amount k.

(4) More generally still, if in the previous model, $sk < N$, we have a model of a queueing system with limited waiting of an amount $N - sk$. There are s servers and batch-wise service of k units per server, as before. The service mechanism is triggered by the presence of k units, not already being served, provided not all s servers are engaged. It is also, again, a model for the re-order level stock control system in which a limited number of demands which cannot be immediately met are held, to be satisfied out of future deliveries.

Finally, $N = \infty$ evidently gives limiting cases of any of the above examples. In some circumstances it may be easier, mathematically, to study the infinite N case directly, and afterwards derive results for the finite N case from those for the infinite case.

4. The Duality Principle

A consequence of the introduction of the two concepts of triggering and of controlled input quantity is that we are enabled to regard what are usually referred to as the "input" and "output" in the case of queues or storage systems, or "demand" and "deliveries" in the case of stocks, as being intrinsically similar in their mechanisms, and we have referred to them both indiscriminately as inputs. This enables us now to consider the basic duality principle in the theory, which, by now, ought to be fairly obvious. We can state it formally as follows.

Consider the general N counter system acted upon by several 0-inputs and 1-inputs which operate according to specified input rules. Then to any realization of this system in which the number of 0's represent the real entities (customers, units of stock, etc.) we can consider a dual realization in which the 1's represent the real entities. Thus a stock model can be the dual of a queueing model with several servers. Or again, a single server queueing model will be self-dual with the roles of inter-arrival and service times interchanged. It is an interesting exercise to consider possible queueing applications of the dual of the cyclical review stock model.

The concept of the dual often provides a useful technique in the mathematical analysis of an actual model. It may turn out that the dual has been treated in the literature, or is in any case the easier to think about. Having carried out the analysis on the dual, the formulae obtained are transformed in an obvious way to apply to the original model. For example, if we calculate for the dual the mean number of items m, then the mean number in the original must be $N-m$.

As mentioned earlier, the limiting case with N infinite may be easier to analyse than the corresponding finite N case. In general, it is clear that if a limiting formula exists for $N = \infty$, it will provide a good approximation for large N. In certain cases, however, it is possible to derive an exact formula for finite N from the infinite case formula. An example of this is provided by Erlang's lost call formula:

$$P_\nu = \frac{\rho^\nu/\nu!}{\sum_{n=0}^{n} \rho^j/j!} \quad (\nu = 0, 1, 2 \ldots N).$$

Here $\{P_\nu\}$ is the equilibrium probability distribution for the number of 1's in the N counter system of the type studied in the first example of Section 3 above. The τ_0 are general independent random variables, and τ_1 is exponential; $\rho = E(\tau_0)/E(\tau_1)$.

Now, we note that this distribution is a conditional one derived from the Poisson distribution,

$$\frac{e^{-\rho}\rho^j}{j!} \quad (j = 0, 1, 2, \ldots).$$

It is easy to show that this is the equilibrium distribution in the infinite N case, and the conditional one for the finite N case could be obtained immediately as a corollary to a general theorem currently being studied by P. D. Finch.

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The concept of duality also applies to waiting times. Consider the time which elapses from the instant when a given counter is switched to 1 until it is again switched to 1, having in the meantime passed through 0. Let us define this as a "total cycle time", c. Now let us define the "waiting time", w_1, as the time which elapses from the instant when the counter is switched to 1 until it is switched to 0. Similarly define the "waiting time", w_0, as the time from 0 to 1. Then clearly we have the duality relation: $w_1 = c - w_0$.

We can apply this idea, for example, to obtaining a model for two dual storage systems. For this purpose, consider an N system with a single 1-input and a single 0-input. Suppose that an input on switching a counter registers upon it the input time which that counter will experience before its next switching. At any instant the sum of all these times registered upon all counters in the same state, together with the unexpired time of the counter which is locked and in process of switching, may be regarded as the size of a store to which inputs of random size are being added at irregular intervals, and on which there is a constant drain.

5. *A Classificatory System*

It will be seen that we have now a broad system of classification for stochastic processes of the type indicated in the above examples: single or multi-channel systems, triggered or untriggered inputs, controlled or uncontrolled input quantities. A wide class of processes will fit naturally into the categories thus defined, and at the same time the categories are sufficiently general not to be in the nature of a strait-jacket, impeding further developments of the theory.

An awareness of these categories will frequently be useful in practical problems in the formulation of the appropriate model. In stock control theory, for example, the distinction between single and multi-channel systems must be made explicit. The triggered or controlled nature of the input rules will often be clear, as for example in the re-order level system where the re-order is put in when and only when the number of items in the system satisfies a certain condition. On the other hand, demand may be operating independently of the state of the system, and if a demand is made when there is a stock-out it is a lost sale. In other stock situations, however, as for example in materials processing, it may be that this output of stock is triggered in that when there is no stock, or more generally when stock has fallen below a minimum batch size, further

processing stops and will re-commence only when more stock becomes available.

In a queueing situation the triggered nature of the service mechanism is not so much in prominence, and the reason is no doubt because in the usual situation if a customer is not present it may be obvious that he cannot be served. This is of course true, but it does mean in fact that the service mechanism is here again triggered by the state of the system—in the simplest case if no-one is there the server is idle and is triggered by the appearance of one unit. On the other hand it is possible for the service to be operating independently of the state of the system—as in the case of a lift or bus which operates whether or not it has a passenger on a particular journey.

Particularly in the case of batch-wise input quantities, it is important to be clear about the input rules.

In some special cases only, the distinction between a triggered and an untriggered input may be unnecessary. Consider, for example, a single input triggered by the presence of one or more units of the opposite kind with input quantity constantly unity, and suppose the input time is exponential. Then, as far as the effect upon the system is concerned, the operation of this triggered input is identical with that of a similar but untriggered one. The reason for this is that, in the case of an exponential distribution, the conditional distribution for the remainder of an input time, given that a certain time has elapsed, is always identical with the unconditional distribution.

Whether or not we care to consider an exponential input as untriggered or triggered would, among other things, depend upon whether we were interested in lost inputs.

6. Conclusion

A general probability model has been described, of which queueing, stock or storage models may all be regarded as special cases. The unifying principle resides in the feed-back feature which these models have in common. Certain general categories have been introduced into which applications can be fitted. Stock control theory in particular would appear to have been impeded in the past through lack of such constructs. It is suggested that economy of thought and further insight will be obtained through use of the neutral terminology introduced in this paper.

The examples given should make it clear that there are diverse ways in which input rules can be constructed. It is in this respect particularly

that I believe the general theory could be enriched. Some of the aims and methods, for example, of control system engineering could well be brought in. For there questions of the efficiency of alternative self-regulatory mechanisms are studied, in addition merely to the calculation of optimum values of parameters in a given system, as is the usual case, for example, in the so-called classical queueing theory.

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The general theory has been formulated here specifically for the study of the operation of man–machine systems. It may well be the case that applications other than those of the type indicated in this paper could be brought into the theory, as, for example, stochastic process models occurring in biology or epidemiology. For the present, however, it is perhaps more profitable to attempt to consolidate mathematically the ground gained so far.

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A Practical Application of Linear Programming in the Mining Industry

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The paper describes the application of linear programming to the allocation of coking coals from twenty-eight collieries to seven central washeries and blending plants in one N.C.B. area. The problem, which is of the simplex type, is reduced to the transportation form. Some details are given of the production method, implementation of the linear programming model and the follow-up.

Introduction

THE problem which is described in this paper is concerned with the transport of coking coals in one of the forty-eight areas of the National Coal Board.

Metallurgical Coke

Coals or mixtures of coals which are used in the coking plants largely determine the quality of coke that is produced. There are two grades of metallurgical coke, viz. foundry coke and furnace coke. Foundry coke is used in the production of special irons and steel and must contain only small quantities of the impurities—sulphur, ash and phosphorus. Furnace coke is used in blast furnaces for the production of pig iron and may contain slightly larger proportions of impurities. In addition, there are close limits on the volatile matter and moisture content of the coals used.

Coals that can be used on their own to manufacture foundry coke are called for convenience in this paper "foundry" coals and the other coals "furnace" coals. Furnace coals can be mixed with foundry coals for foundry coke production.

The limits on impurities are linear; the quantity of impurity in a mixture of coals is the weighted average of the impurities in the individual constituent coals.

Size of Area

The area (approximately 12 miles long by 6 miles wide) which was considered produces about $3{\cdot}1 \times 10^6$ tons per year of coking coal at twenty-eight collieries. Some of these collieries mine more than one quality of coking coal and a total of thirty-seven different qualities are produced

within the area. The impurities content and the coking characteristics differ for each of these coals. In a year about $1{\cdot}7 \times 10^6$ tons of foundry coke and $1{\cdot}4 \times 10^6$ tons of furnace coke mixtures are produced by the area.

The coals are mixed and cleaned at seven central washeries within the area and the resulting washed mixtures are carbonized in nearby coke ovens. At three washeries both furnace and foundry coking mixtures are produced and it may be assumed that there are the equivalent of ten washeries which produce either furnace or foundry mixtures. With the equipment available at the washeries it is difficult to mix coals consistently so that only some high quality furnace coals may be used in the foundry mixtures. Foundry coal is in limited supply and must only be used in foundry mixtures.

Coal is transported from the collieries to the washeries by either British Railways or private road contractors, both of whom charge fixed sums per ton of coal for a particular route. The rail and road charges per ton range from 1s. 7d. to 12s. and 2s. to 6s. 6d. respectively for the available routes. At all the washeries and the majority of collieries the road handling capacity is limited. Rail capacity is also limited at some collieries. The maximum amount of coal which can be transported by road is limited to 13,800 tons per week by the washeries.

Present Planning Method

An estimate of the production of each type of coal and the requirements of the washeries (in order to supply the coke plants) is made for a period of 6 months. These are reduced to average weekly outputs and requirements. An allocation of these average figures is made in the light of previous experience. The many complications which exist make this allocation a difficult one to perform.

This "master allocation" is used as a guide for more exact weekly allocations which are made at the end of a preceding week. Outputs and requirements are estimated each week. If these estimates conflict with the "master allocation" amendments are made for the week. These amendments are made to keep qualities and tonnages at the correct level.

Linear Programming

The Field Investigation Group were asked by the area to determine the minimum transport cost allocation of coals to washeries subject to the following:

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(i) The current washery requirements with regard to quantity and quality of coking coal mixtures.

(ii) The current colliery production schedules.

(iii) Rail and road handling capacities at collieries and washeries. Transport costs were minimized because both washing costs and the revenue from washed coals for any fixed tonnage are constant.

Since the limits on impurities and the transport costs are all linear, the technique of linear programming can be used to discover the minimum cost allocation. In the first instance the problem of the master allocation was considered.

Obviously this problem requires the use of the simplex technique. When the inequalities were written out it was found that there were 740 variables and 171 inequalities. These inequalities are given in the appendix.

The thirty-seven coals are divided into two groups and since foundry coals are only acceptable in foundry mixtures and most furnace coals in furnace mixtures, it is assumed that the cost of sending foundry coals to a furnace mixture is infinite and vice versa. Coals which may be sent to either mixture link the two groups together to make one problem.

The problem now becomes one of transporting coal from thirty-seven different sources by either of two methods to ten destinations. The quality restrictions have been completely relaxed except for the conditions that no foundry coals are allowed to go into furnace mixtures or furnace coals into foundry mixtures. The problem can now be solved by the methods of transhipment[2] which produce a transportation tableau whose dimensions are 131 by 106. Although this is still a large problem it was reasonable to attempt to solve it by hand.[3,4] The amount of calculation is considerably reduced over normal transportation problems of the same size because the only routes used in the transhipment table are the two cost matrices for road and rail transport and the connecting links between them. The diagram given in the appendix shows how the two cost matrices are connected. The restrictions which apply to this modified problem are also indicated.

From observation it was decided to concentrate on road rates and only to consider the road restraints at the collieries. The size has now been reduced to the manageable proportions of forty-eight origins and ten destinations and a large number of routes are "forbidden" because of the quality restrictions. The derivation of this transportation problem is shown in the appendix.

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The solution to the final transportation problem was checked against the various restrictions. In this solution the quality at one washery was suspect and one road restriction had also been broken. By inspection these were corrected and the cost of the corrections was made as small as possible (i.e. for this exercise the extra cost was one penny per ton moved). At this stage the answer which had been obtained was inserted on a transhipment tableau and was found to be the minimum. The time taken for the calculation was approximately 1 hr.

The management accepted the answers and they were implemented as the master allocation. The saving in cost was £20,000 per annum or about 8 per cent of the total transport bill.

Follow-up

After a period of about 3 months the actual weekly allocations were obtained. These had been calculated from the revised master using the original method for amending the master allocation. For each week the optimum was calculated and it was found that the actual allocations could be improved upon by amounts varying from 0 to $2\frac{1}{2}$ per cent. (The average was 1 per cent.) A member of the local staff was taught to perform the calculations. The time involved by the use of linear programming is usually less than by the original methods. The stability of the solution was reasonable since tonnages and demands are fairly constant. When any major change occurs, e.g. opening or closure of a washery, the master allocation is recalculated.

Conclusion

This problem is essentially a straightforward application of a well-known technique. Interest may be felt in this first application of linear programming in the N.C.B. and also the advantages to be gained by approximations.

In this particular problem circumstances and a few short cuts enabled us to solve a very large simplex type problem by transportation and to save a considerable amount of computation.

A very important advantage is that the approximations enabled us to devise a coal allocation procedure which could be operated by the local staff.

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The authors wish to thank the members of the local staff for their help in both defining and understanding the problem. They would also like to thank the N.C.B. for permission to publish this paper. The views expressed in it are their own, and not necessarily those of the National Coal Board.

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Appendix

The simplex problem may be written mathematically as:

$$\text{Minimize} \quad \sum_{i=1}^{37}\sum_{j=1}^{10}\sum_{k=1}^{2} x_{ijk}d_{ijk} \tag{1}$$

$$\text{subject to} \quad \sum_{j=1}^{10}\sum_{k=1}^{2} x_{ijk} = P_i \tag{2}$$

$$\sum_{i=1}^{37}\sum_{k=1}^{2} x_{ijk} = Q_j \tag{3}$$

$$\sum_{j=1}^{10} x_{ijk} \leqslant R_{ik} \tag{4}$$

$$\sum_{i=1}^{37} x_{ijk} \leqslant S_{jk} \tag{5}$$

$$\sum_{i=1}^{37}\sum_{k=1}^{2} a_i x_{ijk} \leqslant A_j \tag{6}$$

$$\sum_{j=1}^{37}\sum_{k=1}^{2} b_j x_{ijk} \leqslant B_j \tag{7}$$

$$\sum_{i=1}^{37}\sum_{k=1}^{2} c_i x_{ijk} \leqslant C_j \tag{8}$$

$$x_{ijk} \geqslant 0 \tag{9}$$

where

x_{ijk} = tonnage of ith coal sent to jth washery by kth method,

a_i, b_i, c_i = ash, sulphur, and volatile matter content of ith coal,

A_j, B_j, C_j = maximum permitted level of ash, sulphur and volatile matter at jth washery,

P_i = total production of ith coal (tons),

Q_j = total requirement of jth washery (tons),

R_{ik} = capacity restriction on ith coal transported by kth method,

S_{jk} = capacity restriction at jth washery on input by kth method,

d_{ijk} = cost of transporting 1 ton of ith coal to jth washery by kth method.

	Colliery 1.....37	Colliery (Road) 1.....37	Washery (Road) 1.....10	Colliery (Rail) 1.....37	Washery (Rail) 1.....10	
Washery 1 ⋮ 20	*M*	*M*	*P*	*M*	*P*	Tonnage required
Colliery (Road) 1 ⋮ 37	*P*	*P*	*M*	*M*	*M*	Colliery road capacities
Washery (Road) 1 ⋮ 10	*M*	Road rates (colliery to washery)	*P*	*M*	*M*	Washery road capacities
Colliery (Rail) 1 ⋮ 37	*P*	*M*	*M*	*P*	*M*	Colliery rail capacities
Washery (Rail) 1 ⋮ 10	*M*	*M*	*M*	Rail rates (colliery to washery)	*P*	Washery rail capacities
	Tonnage available	Colliery road capacities	Washery road capacities	Colliery rail capacities	Washery rail capacities	

Where M is a matrix consisting of Ms (infinite costs) everywhere and P is a matrix with Ms everywhere, except for the leading diagonal, which has zero for each element.

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When the problem is modified to enable the transhipment technique to be applied the inequalities (6), (7) and (8) are ignored. In place of these inequalities certain routes are given an infinite cost to prevent, for example, furnace coals going to a foundry mixture. The layout for the solution by this method is shown in the diagram opposite.

For the second modification we write y_{ij} for the tonnage of the ith coal sent to the jth washery by road and z_{ij} as the tonnage sent by rail. The road restrictions at the washeries are ignored.

Then we minimize $\sum_{i=1}^{37} \sum_{j=1}^{10} (y_{ij} d_{ij1} + z_{ij} d_{ij2})$

subject to

$$\sum_{j=1}^{10} y_{ij} = R_{i1}$$

$$\sum_{j=1}^{10} z_{ij} = P_i - R_{i1}$$

$$\sum_{i=1}^{37} (y_{ij} + z_{ij}) = Q_j$$

$$y_{ij};\ z_{ij} \geqslant 0$$

If we write

$$W_{rj} = y_{ij};\ T_r = R_{i1};\ e_{rj} = d_{ij1} \quad (\text{for } r = 1, \ldots, 37;\ i = 1, \ldots, 37)$$

and

$$W_{rj} = z_{ij};\ T_r = P_i - R_{i1};\ e_{rj} = d_{ij2} \quad (\text{for } r = 38, \ldots, 74;\ i = 1, \ldots, 37)$$

we minimize $\sum_{r=1}^{74} \sum_{j=1}^{10} w_{rj} e_{rj}$

subject to

$$\sum_{j=1}^{10} w_{rj} = T_r$$

$$\sum_{r=1}^{74} w_{rj} = Q_j$$

$$w_{rj} \geqslant 0$$

Since $P_i = R_{i1}$ for twenty-six of the collieries $T_r = 0$ for twenty-six sources and the problem becomes one of transporting from forty-eight origins to ten destinations.

A Method of Dealing with Certain Non-linear Allocation Problems Using the Transportation Technique

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An investigation was carried out to establish a method of determining the allocation between certain factories of a known production requirement so as to minimize the total expense of production. Each factory was made up of a number of departments or services whose total expense varied non-linearly with production level.

The method was to divide the production range of each department in each factory into regions of approximate linear expense, and consider the associated total company expense connected with the optimum allocation for each possible group of regions (one region in each department). In theory the simplex technique could be applied to each group to find its optimum allocation, but as the number of groups is prodigious for quite simple problems this is not practicable. (In the actual problem for which the method was produced there was of the order of 100,000 groups.)

A procedure was devised using the transportation technique together with various lemmas proved in the text and feasibility considerations, by which it was possible to reduce rapidly the number of groups needed to be considered in detail to just a few.

Other problems should be amenable to this method provided they are similar to the one above in two respects.

(1) The departmental variation of expense with production from one region to the next is in general continuous with decreasing rate of change of expense, i.e. the variation is concave.

(2) It should be possible to put the requirements into one of the units of capacity usage in which there are important physical or maximum capacity restrictions.

In addition, the method is particularly suitable when the number of factories is not large.

As the method required repeated applications of the transportation model, a technique was devised for obtaining an optimum solution to this model more quickly than by the usual techniques. This technique will be described in another paper.

An example of a whole problem is given after the Appendix. A full solution of this problem is shown in which occur most of the points dealt with in the text.

The problem for which the method was devised, was one of minimizing the total expense of production of twenty common products between three factories. Each factory had seven departments or services, in most of which two regions were required to describe satisfactorily a department's variation of expense with production, in linear terms. A complete solution was obtained in a few days using the method described without recourse to an electronic computer (not counting the initial computations of costs, etc. required).

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1. Introduction

In planning production between a group of factories, each of which is capable of making some or all of the products required, a problem arises as to which factory should produce what products to minimize total expense, subject to certain policy restrictions, or technical or capacity limitations.

If the total expense of each factory can be expressed as a linear function of the amount of each product it makes, this allocation problem can generally be solved directly by the use of the simplex technique, and in certain cases by the use of the transportation technique.[1, 2] The simplex technique has also been extended to cover some non-linear expense functions, mainly of a convex form.[3, 4]

In the particular problem examined the expense functions were found to be non-linear but in general concave, i.e. continuous with decreasing rate of increase of expense with production. A method was evolved which uses the transportation technique to reduce the problem to one of solving a few linear sub-problems.

2. A General Solution of the Non-linear Allocation Problem

Consider a department or service of a factory in which the total weekly expense can be expressed as a function of some basic unit of capacity usage.

$$T = f(c) \tag{1}$$

where T is the total weekly expense, and c is the numbers of units of capacity used per week. c will be bounded by some physical restriction such as

$$0 \leqslant c \leqslant p \tag{2}$$

where p is the maximum capacity (so many units of capacity per week). T will monotonically increase with c.

It is possible for the purpose of this exercise to divide the range of $c(0, p)$ into *regions* in each of which T can be considered to vary linearly with c. The number of regions needed will depend on the accuracy required. For a given region

$$T = F + Vc \tag{3}$$

where F and V are constants, c being bounded by some constraint such as

$$a \leqslant c \leqslant b \tag{4}$$

a and b, also constants, represent artificial constraints to maintain c in a region where equation (3) is valid. The finite set of regions (a, b) cover the whole of the range $(0, p)$ so that T is completely defined.

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The total weekly expense of each service or department of the factory can be similarly dealt with.

For each department a given region of its capacity range is chosen. This choice is called a *set* of regions. The total weekly expense of the ith department can be written

$$T_i = F_i + V_i c_i \tag{5}$$

valid in the region

$$a_i \leqslant c_i \leqslant b_i \tag{6}$$

Suppose the factory makes a number of different products called $X_1 \ldots X_m$. In a week, x_j of product X_j are made. A unit of product X_j uses n_{ij} units of capacity in the ith department.

$$\therefore c_i = \sum_{j=1}^{m} n_{ij} x_j \text{ for all } i \tag{7}$$

and (5) and (6) become

$$T_i = F_i + \sum_{j=1}^{m} (V_i n_{ij}) x_j \tag{8}$$

$$a_i \leqslant \sum_{j=1}^{m} n_{ij} x_j \leqslant b_i \tag{9}$$

If there are s departments and services and T is the total weekly expense of the factory

$$T = \sum_{i=1}^{s} T_i = \sum_{i=1}^{s} F_i + \sum_{i=1}^{s} \sum_{j=1}^{m} (V_i n_{ij}) x_j$$

Let
$$F = \sum_{i=1}^{s} F_i$$

and
$$k_j = \sum_{i=1}^{s} (V_i n_{ij}) \tag{10}$$

$$\therefore T = F + \sum_{j=1}^{m} k_j . x_j \tag{11}$$

(11) is subject to s restrictions $1 \leqslant i \leqslant s$

$$a_i \leqslant \sum_{j=1}^{m} n_{ij} x_j \leqslant b_i \tag{12}$$

The ranges of capacity usage of departments of other factories can similarly be divided into regions. A region of capacity usage can be chosen for each department of a factory and similar equations to (11) and (12) can be produced for that factory.

The problem is one of allocating a given production requirement between various factories to minimize the sum total expense of production

of all the factories. For each factory involved a given *set* of regions can be chosen and an optimum allocation, if such exists, restricting the capacity of each department to its chosen range can be found by the use of the simplex technique. This would be the one which minimized the sum of all the factory expenses T (equation 11), subject to the production requirements and all the restrictions similar to (12). The minimum total expense and the optimum allocation found are valid only for the particular choice of sets of regions made. These sets of regions (one for each factory) are called a *group* and the minimum total expense and the optimum allocation found for a particular group called the *group minimum expense* and *group optimum allocation* respectively.

If all the group minimum expenses are found, i.e. by considering every possible set of regions of a factory with every other set of regions of each of the other factories, the lowest value of it and the corresponding group optimum allocation would be the required solution. These are called *absolute minimum expense* and *absolute optimum allocation* respectively.

Although in theory the above procedure would give the required solution, the method is of course completely impractical as it stands, even with the use of an electronic computer, due to the effort and time required to evaluate the prodigious number of groups obtained even in relatively simple problems.

However, in certain classes of problems, the number of groups needed to be considered in detail in obtaining the required solution can be logically reduced to just a few by the following considerations.

3. *Reduction of the Number of Groups to be Considered*

(*a*) *Choice of Regions*

The best curve through the information representing the expense variation of departments might be a series of linear portions separated possibly by identifiable steps (additional machines being introduced, etc.) as Figure 1, or alternatively a curved regression line as Figure 2.

For a variation as Figure 1 it would appear that there are six potential regions as numbered.

The variation shown in Figure 1 could be represented by fewer lines with the penalty of some inaccuracy. It would, of course, be possible to put one approximate line through the variation, which although desirable in reducing the number of groups, introduces such inaccuracies as to make the solution obtained of no great value. A compromise should be made

between the number of regions and the accuracy required. In Figure 1 it would be adequate in the first instance to replace the variation between

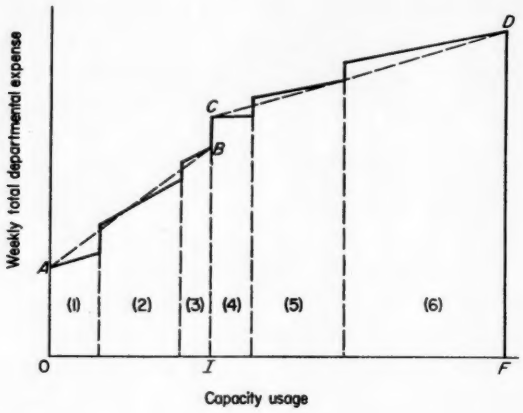

FIGURE 1.

A and B by the mean line AB and to thus amalgamate the regions (1), (2) and (3) and similarly replace the variation between C and D by the mean

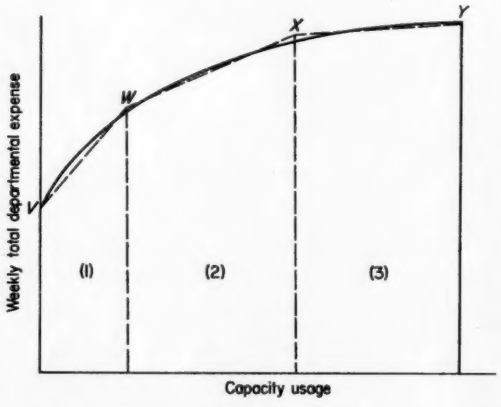

FIGURE 2.

line CD, as in Figure 3. Similarly, the variation in Figure 2 could be replaced by the lines VW, WX, and XY and three regions defined. The proviso on the use of these approximations is

(a) the maximum amount of error introduced is known,

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(b) a solution having been found for a particular group of regions, these regions can be re-defined more accurately and a further solution found.

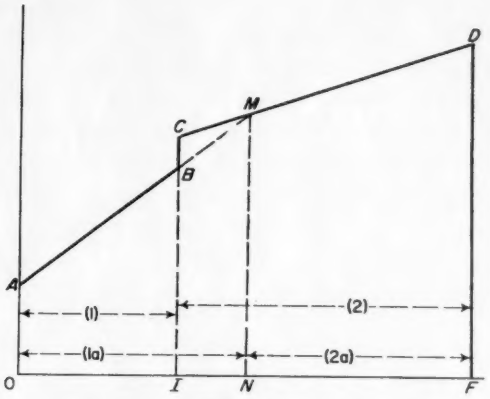

FIGURE 3.

A further simplification can be made with advantage to the variation shown in Figure 3 by producing the line AB until it meets CD in M. The new variation is taken as the lines AM and MD and the new regions as shown as (1a) and (2a). The final variation is in a form which is easily handled by the method, i.e. it is concave, the rate of increase of expense decreasing with capacity usage and no discontinuities.

Suppose the complete problem is solved using the lines AM, MD to represent the variation of expense in Department X.

If the solution is found to put a capacity usage in Department X outside the range IN, the solution will be the same as the one that would be obtained using the line CM instead of BM to represent the variation of expense in region IN. For in the former solution the absolute minimum expense will, by definition, be lower than that corresponding to *any* other allocation; but for any allocation with capacity usage in the range IN, replacing BM by CM increases its total expense. Hence, the absolute minimum expense will be lower than that for any such allocation and thus would be the same if the line CM was used.

In the event of a solution found to require a capacity usage at an inner point of the range IN, a re-appraisal must be made using the variation CM.

It may sometimes be possible to anticipate what the capacity usage in some departments will be for the absolute optimum allocation. In such a department it may be possible to find a linear function of expense over the

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whole range of capacity usage, going through the real value of the expense at the expected value of capacity usage, and such that at any value of the capacity usage the real expense is not less than it, e.g. if the anticipated capacity usage of the department shown in Figure 4 is at Y, the line AYBE would fulfil the above requirements. By a similar argument to that used in the previous paragraph it can be seen that if the complete solution using this function gives the departmental capacity usage at its anticipated value, this solution will be the same as the one that would be obtained using the correct departmental expense variation, i.e. if the solution found using AYBE as the variation gave any point of AB as the capacity of this department, the solution would be the same as if the correct variation had been used. If some capacity usage outside AB is found, the solution is invalid, but this does not disprove that the capacity usage of the correct solution lies in AB.

The above procedure may help to reduce the number of regions in general, and could be of particular use in dealing with convex variations similar to Figure 4, in those cases where a solution is expected *within* or at a physical end point of a region, for example, if the anticipated capacity usage is somewhere in the region defined by AB *other than* B.

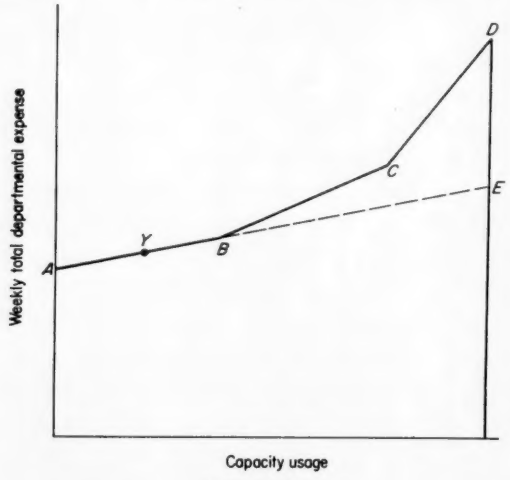

FIGURE 4.

(*b*) *Feasibility Restrictions*

The regions of each department are now presumed uniquely defined. The number of combinations possible can be reduced by two types of feasibility considerations:

(i) In a particular factory the choice of a region of capacity usage in one department somewhat restricts the choice of regions for the others as level of production in one department is to some extent controlled by the level of production in another.

The reduction of possible combinations of regions by this means need not be done precisely taking into account actual requirements. If the n_{ij} for a given i are not too dissimilar, consideration of the equations (derived from equation 7):

$$[\min_j (n_{ij})] \sum_{j=1}^{m} x_j \leqslant c_i \leqslant [\max_j (n_{ij})] \sum_{j=1}^{m} x_j \quad \text{for all } i \tag{13}$$

shows incompatibility of regions easily and still keeps the problem general (i.e. free from particular requirements).

(ii) In the same department of each of the factories, introduction of the total requirements will also somewhat restrict the choice of regions of capacity usage in these departments. A given choice of region in this department of each factory must be compatible with the requirements.

Here, if n_{ij}, for a given i and j, is not too dissimilar from factory to factory, by finding its minimum and maximum values, upper and lower bounds can be put on the total number of units of capacity of department i, the requirements could possibly occupy. These can be obtained easily and used to show incompatibility of regions within corresponding departments of each of the factories.

(c) *Local Optima*

An allocation, such that *any* small change in allocation physically possible would produce an increase in total expense, is called a *local* optimum allocation. The absolute optimum allocation must also be a local optimum. Hence, the least of the total expenses associated with the local optima will be the absolute minimum total expense and the associated allocation the absolute optimum allocation.

By the following considerations these local optima can be relatively easily isolated. If their number is not very large the allocation and total expense associated with each can be evaluated and the required absolute solution found. The number will not be large when the variation of departmental expense, from one region to the next of higher capacity usage, is in general continuous and at a lower rate of increase (i.e. the variation is concave).

The transportation technique is used to show whether a local optimum is included in a particular choice of group of regions. Thus, having chosen

a group of compatible regions, the costs k_j can be found for each factory. The transportation table is set out with these costs inserted (see Table 1). Let the cost of product j at factory A be k_{jA}. The requirements are inserted on the bottom column. The minimum value of k_{jA}, k_{jB}, k_{jC} etc. is ringed (say k_{jE}), and all of product J is sent to factory E. All the allocation is made in this manner.

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TABLE 1

Product / Factory	Product 1	2	3	..	..
A	k_{1A}	k_{2A}	k_{3A}	..	..
B	k_{1B}	k_{2B}	k_{3B}	..	..
C	k_{1C}	k_{2C}	k_{3C}	..	..
	..	..	..	..	..
Requirements					

Now if this allocation gives capacity usage of every department within the region chosen originally this is a local optimum allocation and the allocation should be registered and its total expense evaluated. For within the costs appropriate to this group of regions, each requirement has been made as cheaply as possible and thus the allocation gives the minimum total expense for all allocations *within* the group. Since the allocation was made without reference to the bounds of the regions the occurrence of the capacity usage of any department exactly on a bound of a region is a matter of chance only. Thus each capacity usage is a finite distance from either bound, so that *any* small change in allocation by less than some given small finite amount will keep each capacity usage within its appropriate region. Hence, this allocation is a local optimum. (If the capacity usage did fall on a bound the allocation could be treated as if it were a local optimum.)

Nearly always in this\first allocation there will be at least one capacity usage outside its specified region, and in general this will be obvious from feasibility consideration as in b(i) without resort to calculation of actual capacity usage. The bound of a region which has been violated can either be:

(1) An artificial restriction dividing regions to ensure that the capacity usage of a department remains within a region consistent with the costs k_j. The two regions separated will either have a variation of expense which is (a) concave, or (b) convex or discontinuous.

(2) A physical limitation, such as 100 per cent capacity level, or a restriction enforced on the solution such that the capacity usage of a given department should not exceed or be less than a given amount.

The course of action to be followed next depends on the number and type of restrictions broken.

Depending on the restrictions broken, it may be possible to say immediately that the group considered could not contain a local optimum and hence need not be considered further. It may be also possible to say that several other groups could not contain a local optimum.

It may be necessary to take into account in the transportation model some of the broken restrictions, i.e. re-allocate to find the optimum allocation with these restrictions enforced. Depending on this new allocation and whether it breaks any further restrictions, and if so their nature, one or other of the possibilities listed here will be found to apply.

It may be possible, by relaxing some restriction or other, to be able to say that a local optimum is included in the group as considered originally, but with somewhat extended regions. The optimum allocation in this case will have been obtained simply from the transportation model. The corresponding total expense is then assessed. Then if in subsequent investigations a valid local optimum is found in some other group with expense less than this, the former group need not be considered further.

As a last resort one may admit the possibility that a local optimum may be contained in the group and that it can only be found by use of the simplex. If many such groups are found, the method described herein is obviously unsuitable for the problem concerned. However, if it is similar to the original problem for which the method was devised, in the respects summarized in the conclusion, only a few such groups should be found.

A statement of the principles governing the above decisions is given in the next section. The proof of the lemmas underlying this statement is given in the Appendix.

It is recommended to obtain an understanding of how these principles are applied and the method by which the number of groups needed to be considered is reduced, that the example given at the end of this paper be read through. It was found much easier to describe the application of the method by the aid of this example than by a formal statement.

(*d*) *The Principles of the Method*

(i) If there is just one department in all the factories whose capacity usage is put outside its specified region by the allocation described in (c)

and the bound thus broken is of the type (1a), then the group of regions concerned does not contain the absolute optimum allocation. Further, this group does not contain a local optimum.

This statement, which is proved in the Appendix, used in various ways, is the main instrument of the method. Obviously if the statement as it stands is found to apply to a particular group, this group or combination of regions need not be considered further.

(ii) Further to (i) consider the department which contains the broken restriction. If there are several regions between the one chosen for the group and the one in which the capacity usage is put by the allocation of (c) and if each of these regions is separated by a bound of the type (1a), then it can be said that the group containing any of these regions substituted for the original does not contain a local or absolute optimum.

(iii) If there are several departments of the various factories whose capacity usages are put outside their specified regions and the bounds thus broken are all of the type (1a), then the group of regions concerned does not contain the local or the absolute optimum. Further, in many instances it may be possible at this stage from this information to eliminate several other groups by the reasoning explained in the Appendix.

(iv) If there are one or more bounds broken of the type 2 or (1b) then those bounds which can be expressed in the same units as the requirements can be enforced in the normal manner in a new allocation within the transportation model. This resulting allocation is then examined for further broken restrictions in the same manner as was the original allocation.

(v) Any bounds broken of the type 2 or (1b) to which (iv) does not apply are ignored or relaxed in the first instance. This is equivalent to an artificial extension of the appropriate region in one direction.

For local optima found after the application of (v) each associated total expense is simply calculated and noted. Any found with a total expense less than £X can be ignored as such a group could not contain the absolute optimum; where £X is the lowest value of the total expense of the valid local optima (i.e. those found without invoking (v)).

If £X is found to be lower than all total expenses derived from local optima found after the use of (v), then it is the absolute minimum total expense and its associated allocation is the absolute optimum and the problem is solved. Otherwise the local optima found using (v) with a lower associated total expense than £X need to be investigated further, possibly with the use of the simplex technique.

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If there obviously are restrictions in one particular unit of capacity usage which tend to dominate the solution, i.e. are the most severe, then unless this unit is that of the requirements the method described here will not be satisfactory. However, it may be possible to put the requirements in terms of this unit by approximation or standardization. (Such methods for the ordinary transportation technique are described elsewhere.[5]) In such a case the problem can be satisfactorily dealt with by the present method.

4. *Actual Problem Solved*

The problem for which the method described above was devised was one of finding the minimum total expense of allocating twenty common products between three factories. Each factory had seven departments and services in most of which two regions were required to describe satisfactorily, in linear terms, the departmental variation of expense. In the first instance then, there were about 100,000 possible groups. This was reduced to about 100 by feasibility considerations.

The number of groups needed to be considered in detail was reduced to four with the use of the transportation model and the above principles, and the optimum was found without recourse to the simplex technique.

As obtaining this solution required repeated applications of the transportation model, a technique was devised for obtaining an optimum solution to this model more quickly than by the usual techniques. This technique will be described in another paper.

5. *Conclusions*

The method discussed will obtain the optimum solution required in those non-linear allocation problems which have the following characteristics:

(1) The departmental variation of expense with production from one region to the next is in general continuous with decreasing rate of change of expense, i.e. the variation is concave.

(2) It should be possible to put the requirements into one of the units of capacity usage in which there are important physical or maximum capacity restrictions.

In addition the method is particularly suitable when the number of factories is not large.

Where the use of the method is appropriate the solution will be obtained relatively rapidly.

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The investigation and solution of the actual problem for which the method was developed was done as a joint study by Mr. James and the author.

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Appendix

Lemma 1

If the optimum allocation without restrictions derived from the costs pertaining to a given group of regions breaks a bound of only one region and this bound separates regions over which the variation of expense is concave, the given group does not contain the absolute optimum allocation.

Suppose the department concerned has a variation of total expense as shown in Figure 5, and region 2 is the region taken for this department in the group considered, i.e. the line AD has been assumed to represent the variation of expense of this department. Also suppose the capacity usage of this department given by the optimum allocation without restrictions, using the transportation model, is Y_1 outside region 2. (The following arguments are similarly applicable if Y_1 occurred in region 1.)

This allocation has an associated total expense less than any other based on the same costs. If the same allocation were made for the costs derived from the group of regions as before, but with region 3 substituted for region 2, the total expense of every other department would be exactly the same as before, as the same amounts of each requirement are being made at each factory. In the department concerned, the total departmental expense would be Z_1Y_1 compared to X_1Y_1 previously. Thus there is *an* allocation in the group containing region 3, giving a lower total expense than the *optimum* for the group containing region 2 *without* regional constraints.

Therefore *the optimum allocation for the group containing region 3 has a lower total expense than the optimum for the group containing region 2.* Thus the latter group of regions need not be considered further as it could not contain the absolute optimum.

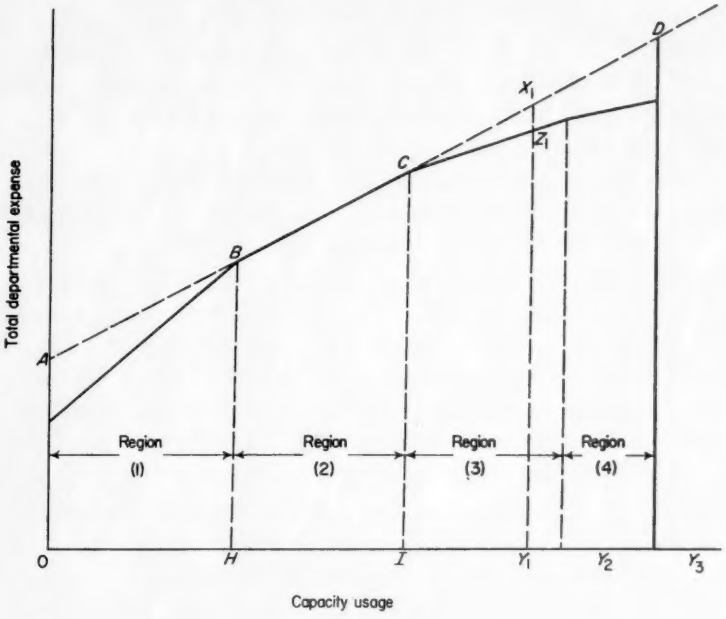

FIGURE 5.

The above conclusion is still valid even if the capacity usage is outside the next region, e.g. at Y_2 or Y_3. This is shown by the following lemma:

Lemma 2

If the optimum allocation found without restrictions puts the capacity usage of Department X at $\bar{c}$, and c_1 and c_2 are two values of capacity usage in this department such that $c_1 < c_2 < \bar{c}$, then

$$E_{c_1} > E_{c_2} > E_{\bar{c}}$$

E_{c_1} is the total expense associated with the *optimum* allocation under the restriction that the capacity usage of Department X must be kept less than or equal to c_1.

Further, this optimum allocation with associated expense E_{c_1} has capacity usage in Department X at c_1.

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This can be proved as follows:

In the original allocation without restrictions, the factory whose department is under consideration was allocated the complete requirements of a number of products. Suppose the factory is A and the products are 1, 2 and 3. These products can only be made elsewhere at additional cost per item of product. (This assumes that for each j; k_{jA}, k_{jB} etc., are all different. If some of them are the same the lemma needs to be slightly modified.) These products represent a capacity usage of $\bar{c}$ for Department X. If the department is at any capacity usage less than $\bar{c}$ not all of products 1, 2 and 3 can be made at factory A.

Suppose there is some value of capacity usage, say c_4, such that $c_4 < \bar{c}$ and the optimum allocation restricting the capacity usage of Department X to be less than or equal to c_4, puts capacity usage at some value c_5 other than c_4. Then as $c_5 < \bar{c}$ there are some of the products 1, 2 or 3 made other than at factory A. Suppose some of product 1 is being made at factory B.

Consider the optimum allocation with capacity usage at c_5. If some of product 1 is moved from B to A, there will be a net reduction in total expense. Although the capacity usage of the department, c, will increase, the amount can be chosen sufficiently small to keep $c_5 < c < c_4$. Hence there cannot be an optimum allocation with capacity usage at $c_5 < c_4$. Therefore the optimum allocation for $c \leqslant c_4$ has a capacity usage of c_4 and is unique.

Thus if $c_1 < c_2 < \bar{c}$, E_{c_2} has capacity usage at c_2 and is less than the total expense associated with any other capacity usage $c < c_2$.

i.e. $$E_{c_1} > E_{c_2}.$$

Similarly, $$E_{c_1} > E_{c_2} > E_c.$$

N.B.—In lemmas 1 and 2 it was assumed for simplicity that the optimum allocation was made without any restrictions whatsoever. However, the lemmas are equally true if the optimum allocation had been made with some restrictions enforced.

Applications of Lemmas 1 *and* 2

Suppose the capacity usage of Department X was found without restrictions to be at Y_2 or Y_3 (Figure 5) for the costs obtained from the group containing region 2. From lemma 2 it follows that if Y is any value of capacity usage within region 3, there will be an allocation with capacity usage at Y which will have a total expense less than that associated with any allocation with capacity usage within region 2.

Thus from lemma 1 it follows that the group chosen containing region 2 could not contain the absolute optimum allocation, and thus need not be further considered.

In addition, if region 3 and region 4 are separated by a restriction of the type (1a), the group containing region 3 instead of region 2 also need not be considered further. For the only effect on the costs in Table 1 of substituting region 3 for region 2 will be to reduce all those belonging to the factory whose department is under discussion. Thus, those products for which this factory had the lowest costs in the original table will still be the lowest in the new table, and all the products previously allocated to this factory in the allocation without restriction will still be allocated to it; other products might also be allocated there. Hence, the capacity usage of the department will be not less than before, and so lemma 1 or lemma 2 could be applied to show there is no need to consider this group.

In the event of the original capacity usage being Y_2 or Y_3, region 4 could be substituted for region 2 and a new allocation made.

If in the former case (Y_2) the capacity usage now found is within region 4, this allocation is a local optimum and the total expense evaluated. More probably, the unrestricted capacity usage will now be found to be outside of region 4, i.e. outside a physical limitation. This is considered in the text.

The optimum allocation restricting the capacity usage of Department X to region 2 will by lemma 2 put the capacity usage on region 2's upper bound at I. Consider this optimum allocation. Let Y be some capacity usage greater than I. Then from lemma 2, no matter how near Y is to I, $E_{C_Y} < E_{C_I}$. But as in lemma 1, if region 3 is substituted for region 2, the same allocation with capacity at Y will have an associated total expense less than E_{C_Y} and thus less than E_{C_I}. This implies that the optimum allocation found with capacity usage at I is not a local optimum. Hence, the group of regions considered cannot contain a local optimum allocation.

Consider the case mentioned in the text where several bounds all of the type (1a) are broken in the original allocation. If a new optimum allocation is made without restrictions except for these broken bounds, it follows that the capacity usage of at least one of the departments whose bound has been broken must be at the value of the bound. (This can be seen by assuming this statement is not true. There will obviously be some change in the allocation from the new optimum allocation, returning it towards the original optimum allocation which, if it is kept small enough, will satisfy the restrictions and will also result in a net reduction in total

expense.) Hence from the above it follows that the group concerned does not contain a local optimum.

The effect of replacing the appropriate adjacent region of a department for the region whose bound has been broken, has been considered above. In the case where there are several bounds broken the effect of this change in the other departments, in general, is that a capacity usage put higher than its appropriate region for the original optimum allocation will now be made even higher. In cases where this will obviously occur it is thus possible to discard a number of groups which could not contain a local optimum, e.g. suppose two of the bounds broken were in departments A and B respectively of the same factory. Suppose region 1 of department A and region 1 of department B had been taken in the original group. Substituting region 2 for region 1 of department A but keeping all other regions of the group the same, the only effect on an optimum allocation without restrictions will be that possibly more products will be allocated to the factory concerned. Thus this allocation will still keep the capacity of department B beyond its region 1. Hence the new group cannot contain a local optimum allocation.

Similar examples can often be seen without further investigation.

An Example of the Solution of a Non-linear Production Allocation Problem

Statement of the Problem

Three factories A, B and C can each make any of the products 1, 2 and 3.

TABLE 2

MACHINE WEEKS REQUIRED BY A PRODUCT AT A GIVEN FACTORY

Product \ Factory	A	B	C
1	$\frac{1}{2}$	$\frac{1}{2}$	1
2	1	1	1
3	2	2	$2\frac{1}{2}$

Units of Capacity Usage. A factory consists of two departments:

Department "a" where some sort of raw material is processed. Capacity usage in this department is in tons per week of output and the products

can be put in terms of the same unit, i.e. a ton of any product requires a ton from any one of the three factories' processing departments.

Department "b" where the processed material is "machined" in some way. The basic unit of capacity usage here is a "machine week" equivalent to one machine being occupied for a week. A ton of each product requires a different amount of machine weeks as given by Table 2.

Requirements. There are weekly requirements for:

51 tons of product 1
41 „ „ „ 2
11 „ „ „ 3

Capacity Restrictions.—There are only a given number of machines and a given processing capacity in each factory. These are at:

Factory A—45 machines and 50 tons per week
„ B—60 „ „ 40 „ „ „
„ C—50 „ „ 50 „ „ „

Variation of Expense.—The variation of total departmental expense for each department has been determined and regions of linearity decided. These are shown on the following diagrams and summarized in Table 3 ("t" and "m" refer to the number of tons and machine weeks occupied respectively).

TABLE 3
REGIONS OF LINEARITY AND APPROPRIATE VARIATION OF EXPENSE

Factory	Department	Region	Range of Region	Variation of Total Departmental Expense £
A	a	1	0–20 tons	(100+10t)
		2	20–50 tons	(200+5t)
	b	1	0–20 m/weeks	(50+5m)
		2	20–45 m/weeks	(130+m)
B	a	1	0–40 tons	(10+7t)
	b	1	0–19 m/weeks	(73+6m)
		2	19–40 m/weeks	(130+3m)
		3	40–60 m/weeks	(170+2m)
C	a	1	0–40 tons	(50+9t)
		2	40–50 tons	(170+6t)
	b	1	0–15 m/weeks	(60+3m)
		2	15–50 m/weeks	(80+3m)

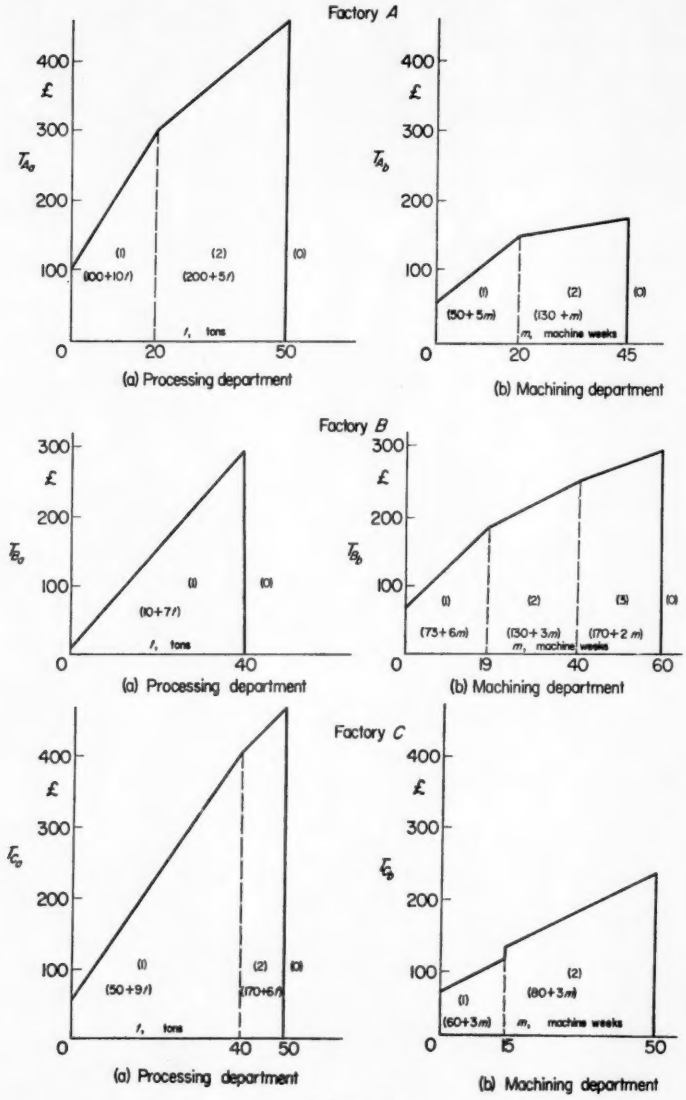

FIGURE 6.

Notation

B_a refers to department "a" of factory B.

$30B_a$ means that the capacity usage of B_a is 30 units; in this case the units are tons.

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$[1]B_b$ means that the capacity usage of B_b is within region [1].

$[0]A_a$ means that the capacity usage of A_a is greater than the physical maximum quoted (i.e. 50 tons).

$[1]A_a \sim [1]A_b$ means that $[1]A_a$ and $[1]A_b$ are feasible.

$[2]C_a \nsim [1]C_b$ means that $[2]C_a$ and $[1]C_b$ are not feasible.

$[1]A_a \sim [1]B_a \nsim [1]C_a$ means that $[1]A_a$ and $[1]B_a$ are not feasible with $[1]C_a$.

$[1]'B_b$ means that the capacity usage of B_b can have any value, but the appropriate total expense of B_b varies as in $[1]B_b$.

$[1]/[2]C_a$ refers to the restriction between $[1]C_a$ and $[2]C_a$.

L.O. means local optimum.

A.O. means absolute optimum.

L.O. $\in ([1]A_a \sim [1]A_b \sim [1]B_a \sim [1]B_b \sim [1]C_a \sim [1]C_b)$ means that for the costs appropriate to this choice of regions there is an allocation which is a local optimum, such that the capacity usage of each department lies within the region chosen.

L.O. $\notin$ ... means that there is no local optimum within the group chosen.

Solution

Feasibility Restrictions

Factory Wise

Factory A	$[1]A_a \sim [1]A_b$	e.1
	$[1]A_a \sim [2]A_b$	e.2
	$[2]A_a \sim [1]A_b$	e.3
	$[2]A_a \sim [2]A_b$	e.4
Factory B	$[1]B_a \sim [1]B_b$	e.5
	$[1]B_a \sim [2]B_b$	e.6
	$[1]B_a \sim [3]B_b$	e.7
Factory C	$[1]C_a \sim [1]C_b$	e.8
	$[1]C_a \sim [2]C_b$	e.9
	$[2]C_a \sim [2]C_b$	e.10

e.1 is taken to refer to the "costs" of the various products if made at factory A within region [1] of A_a and region [1] of A_b.

N.B.

$[2]C_a \nsim [1]C_b$ as 40 tons require 40 machine weeks at least in factory C.

Department Wise

Department "a"

There is a requirement for a total of 103 tons of products

$$[1]A_a \sim [1]B_a \sim [2]C_a \qquad \text{f.1}$$

$$[2]A_a \sim [1]B_a \sim [1]C_a \qquad \text{f.2}$$

$$[2]A_a \sim [1]B_a \sim [2]C_a \qquad \text{f.3}$$

f.1 is taken to refer to the choice of region [1] of A_a together with [1] of B_a and [2] of C_a.

N.B.

$$[1]A_a \sim [1]B_a \nsim [1]C_a$$

Department "b"

The requirements are for at least $(\frac{1}{2} \times 51) + (1 \times 41) + (2 \times 11) = 88{\cdot}5$ machine weeks and at the most 108·5 machine weeks.

$$[1]A_b \sim [1]B_b \sim [2]C_b \qquad \text{g.1}$$

$$[1]A_b \sim [2]B_b \sim [2]C_b \qquad \text{g.2}$$

$$[1]A_b \sim [3]B_b \sim [1]C_b \qquad \text{g.3}$$

$$[1]A_b \sim [3]B_b \sim [2]C_b \qquad \text{g.4}$$

$$[2]A_b \sim [1]B_b \sim [2]C_b \qquad \text{g.5}$$

$$[2]A_b \sim [2]B_b \sim [1]C_b \qquad \text{g.6}$$

$$[2]A_b \sim [2]B_b \sim [2]C_b \qquad \text{g.7}$$

$$[2]A_b \sim [3]B_b \sim [1]C_b \qquad \text{g.8}$$

$$[2]A_b \sim [3]B_b \sim [2]C_b \qquad \text{g.9}$$

N.B.

$$[1]A_b \sim [1]B_b \nsim [1]C_b$$

$$[1]A_b \sim [2]B_b \nsim [1]C_b$$

$$[2]A_b \sim [1]B_b \nsim [1]C_b$$

Taking into account $[2]C_a \nsim [1]C_b$ there are $(3 \times 9) - (2 \times 3)$ combinations $= 21$ combinations compared to a possible $(2 \times 2) \times (1 \times 3) \times (2 \times 2) = 48$ combinations ignoring feasibility restrictions. All the combinations are systematically worked through.

Unit Costs. First, we obtain the appropriate "costs" for e.1, e.2, etc. The cost per item of a product represents the increase in total expense if an additional item of the given product is made. (These costs are the k_j referred to earlier.)

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e.g. For e.1, additional items cost £10 per ton and £5 per machine week. Thus 1 ton of product 1 costs £10 + ½ £5 = £12½.

TABLE 4
COST IN £

Factory		Product 1	Product 2	Product 3
A	e.1	12½	15	20
	e.2	10½	11	12
	e.3	7½	10	15
	e.4	5½	6	7
B	e.5	10	13	19
	e.6	8½	10	13
	e.7	8	9	11
C	e.8	12	12	16½
	e.9	12	12	16½
	e.10	9	9	13½

e.1 is written (12½, 15, 20) etc.

Reduction of Groups. A start is made with f.1 ~ g.1.

f.1 ~ *g*.1

$$\left.\begin{array}{ll} [1]A_a \sim [1]A_b & \text{e.1} \\ [1]B_a \sim [1]B_b & \text{e.5} \\ [2]C_a \sim [2]C_b & \text{e.10} \end{array}\right\} \text{or f.1} \sim \text{g.1}$$

The normal cost tableau is constructed. Quantities allocated are inserted above the cost and ringed. The last column is for physical restrictions

TABLE 5
COST TABLEAU

Factory \ Product		1	2	3	Physical Restrictions (tons/machine weeks)
A	e.1	12½	15	20	
B	e.5	10	13	19	
C	e.10	[51] 9	[41] 9	[11] 13½	
Requirements in tons		51	41	11	

which are enforced. In the first place allocation is just made to the lowest cost in each column, ignoring *all* restrictions.

In the above allocation two restrictions are broken, [2]/[0]C_a and [2]/[0]C_b. The former is enforced and a re-allocation made.

The technique used throughout this example to find the optimum allocations is one especially devised for obtaining rapid solutions. It will be described in another paper. The method starts by constructing a cost difference table by subtracting the lowest cost in each column of the cost tableau from every other cost in the column. In the present example, optimum allocations can be found almost immediately from this.

TABLE 6
COST-DIFFERENCE TABLEAU

Product / Factory	1	2	3	Physical Restrictions (tons)
A	$3\frac{1}{2}$	6	$6\frac{1}{2}$	
B	[51] 1	[2] 4	$5\frac{1}{2}$	
C	0	[39] 0	[11] 0	50
Requirements in tons	51	41	11	

There are now three broken restrictions,

(1) [2]/[0]C_b as $39+(11\times 2\frac{1}{2}) = 66\frac{1}{2}$ machine weeks are occupied in factory C.

(2) [1]/[0]B_a as 53 tons have been put in factory B.

(3) [1]/[2]B_b as $(51\times\frac{1}{2})+2 = 27{\cdot}5$ machine weeks are occupied in factory B.

(1) is dealt with by replacing [2]C_b by [2]'C_b; (f.1 ~ g.1)' refers to [1]A_a ~ [1]A_b ~ [1]B_a ~ [1]B_b ~ [2]C_a ~ [2]'C_b. Now if there is a L.O. $\in$ (f.1 ~ g.1) the same allocation with the same associated total expense can be made in (f.1 ~ g.1)'.

(2) can be ignored for this allocation, as if the capacity usage of region B_b had been restricted to region [1] the violation [2]/[0]B_a could not have occurred, as [1]B_b ~ [0]B_a.

(3) is an artificial concave restriction and is now the only restriction broken.

Thus, L.O. $\notin$ (f.1 ~ g.1)′ (*N.B.* There still might be a L.O. $\in$ f.1 ~ g.1.)

The next group naturally to be considered is found by replacing [1]B_b by [2]B_b, or e.5 by e.6 ($8\frac{1}{2}$, 10, 13) in Table 4, and obtaining (f.1 ~ g.2)′.

f.1 ~ *g*.2

Physical restrictions of 40 and 50 tons on B_a and C_a are enforced. The optimum allocation is found as above from cost differences. The result is shown in Table 7.

TABLE 7

Factory \ Product		1	2	3	Physical Restrictions (tons)
A	e.1	[13] $12\frac{1}{2}$	15	20	
B	e.6	[29] $8\frac{1}{2}$	10	[11] 13	40
C	e.10	[9] 9	[41] 9	$13\frac{1}{2}$	50
Requirements		51	41	11	

The optimum allocation made above breaks no regional restrictions of (f.1 ~ g.2)′. Further, it breaks no restrictions of f.1 ~ g.2.

$\therefore$ L.O. $\in$ f.1 ~ g.2

Now the associated total expense of this allocation is less than that for any other allocation in f.1 ~ g.2, (f.1 ~ g.2)′, or (f.1 ~ g.1)′ and thus for any in f.1 ~ g.1. (Follows from Appendix; paragraph 4 of lemma 1.)

$\therefore$ A.O. $\notin$ f.1 ~ g.1.

The total expense of the L.O. $\in$ f.1 ~ g.2 = £(100 + 50 + 10 + 130 +

+ 170 + 80) + (13 × $12\frac{1}{2}$) + (29 × $8\frac{1}{2}$) + (9 × 9) + (41 × 9) +

+ (11 × 13) = £1,542.

f.1 ~ *g*.3

f.1 ~ *g*.4

Replacing $[2]B_b$ by $[3]B_b$ or e.6 by e.7 (8, 9, 11) in Table 7 can be seen to give exactly the same allocation shown in Table 7.

Thus, $[3]/[2]B_b$ only is broken, and L.O. $\notin$ f.1 ~ g.4.

f.1 ~ *g*.5

i.e. $[1]A_a \sim [2]A_b \sim [1]B_a \sim [1]B_b \sim [2]C_a \sim [2]C_b$ with associated costs,

A e.2 (10½, 11, 12)

B e.5 (10, 13, 19)

C e.10 (9, 9, 13½)

The optimum allocation restraining factory C to 50 tons gives,

(0, 0, 11) tons to A (i.e. 0 tons of product 1 or 2, and 11 tons of product 3 to factory A),

(42, 0, 0) tons to B

and (9, 41, 0) tons to C.

There are two restrictions broken $[1]/[2]B_b$ and $[2]/[0]B_a$. As $[0]B_a \sim [1]B_b$ our only concern is with the former. L.O. $\notin$ f.1 ~ g.5.

$[1]B_b$ is replaced by $[2]B_b$ or e.5 by e.6 (8½, 10, 13) to obtain.

f.1 ~ *g*.7

Physical restrictions of 40 and 50 tons on B_a and C_a are enforced.

TABLE 8

Factory \ Product		1	2	3	Physical Restrictions (tons)
A	e.2	[2] 10½	11	[11] 12	
B	e.6	[40] 8½	10	13	40
C	e.10	[9] 9	[41] 9	13½	50
Requirements		51	41	11	

There are now no restrictions broken.

Thus, a L.O. $\in$ f.1 ~ g.7.

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However, there is an increase in total expense between this allocation and the optimum found in f.1 ~ g.2 of

$$£(130–50) – (11 \times 1) – [(2 \times 2) + (11 \times 4)] = £21.$$

$\therefore$ A.O. $\notin$ f.1 ~ g.7.

f.1 ~ *g*.6

f.1 ~ *g*.8

f.1 ~ *g*.9

$[2]B_b$ is replaced by $[3]B_b$ in f.1 ~ g.7. e.6 is replaced by e.7 (8, 9, 11). The optimum allocation is exactly as is f.1 ~ g.7. Thus, only $[3]/[2]B_b$ is broken and L.O. $\notin$ f.1 ~ g.9.

f.2 ~ *g*.1

$[2]A_a \sim [1]A_b \sim [1]B_a \sim [1]B_b \sim [1]C_a \sim [2]C_b$ with associated costs:

A	e.3	(7½, 10, 15)
B	e.5	(10, 13, 19)
C	e.9	(12, 12, 16½)

It can be seen that the first allocation, without restrictions puts all products into factory A, breaking in the first place $[1]/[2]A_b$, $([1]A_b \sim [0]A_a)$ $([1]A_b \sim [1]B_b \sim [1]C_b)$.

$\therefore$ L.O. $\notin$ f.2 ~ g.1

$\therefore$ $[1]A_b$ is replaced by $[2]A_b$.

f.2 ~ *g*.5

e.3 is replaced by e.4 (5½, 6, 7) and factory A restricted to 50 tons.

The optimum allocation then gives,

(0, 39, 11) tons to A

(51, 0, 0) tons to B

and (0, 2, 0) tons to C.

Now there are 4 restrictions broken in this allocation:

(1) $[2]/[0]A_b$ as $39 \times 1 + 2 \times 11 = 61$ machine weeks have been allocated to A.

(2) $[2]/[1]C_b$ as only 2 machine weeks have been allocated to C.

(3) $[1]/[0]B_a$ as 51 tons were allocated to B.

(4) $[1]/[2]B_b$ as 25·5 machine weeks have been allocated to B.

(1) is dealt with by replacing $[2]A_b$ by $[2]'A_b$ and (f.2 ~ g.5)″ refers to $[2]A_a \sim [2]'A_b \sim [1]B_a \sim [1]B_b \sim [1]C_a \sim [2]C_b$.

Neither (2) nor (3) can occur if $[1]B_b$ is respected as

$$[1]B_b \sim [0]B_a$$

$$\text{and } [1]B_b \sim [1]C_b$$

Thus, L.O. $\notin (f.2 \sim g.5)''$ and $[1]B_b$ is replaced by $[2]B_b$ to give:

$f.2 \sim g.7$

e.5 is replaced by e.6 ($8\frac{1}{2}$, 10, 13) and physical restrictions of 40 and 50 tons on B_a and C_a are enforced.

TABLE 9

Factory \ Product		1	2	3	Physical Restrictions
A	e.4	[11] $5\frac{1}{2}$	[28] 6	[11] 7	50
B	e.6	[40] $8\frac{1}{2}$	10	13	40
C	e.9	12	[13] 12	$16\frac{1}{2}$	
Requirements		51	41	11	

This optimum allocation was found as the others but serves as a particularly simple example. First take column cost differences.

TABLE 10

Factory \ Product	1	2	3	Physical Restrictions
A	0	0	0	50
B	3	4	6	40
C	$6\frac{1}{2}$	6	$9\frac{1}{2}$	
Requirements	51	41	11	

Now as all costs of C are greater than costs in either A or B,

50 tons will be allocated to A

40 „ „ „ „ „ B

and 13 „ „ „ „ „ C exactly.

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Thus, we take row cost differences by subtracting the lowest row cost in Table 10 from all others in the row.

Allocation is made to "zeros" in the table in the first instance. In this case an allocation can immediately be found allocating products to "zeros" only. Thus, this must be the optimum allocation.

TABLE 11

Factory \ Product	1	2	3	Total Amount Made
A	[11] 0	[28] 0	[11] 0	50
B	[40] 0	1	3	40
C	$\frac{1}{2}$	[13] 0	$3\frac{1}{2}$	13
Requirements	51	41	11	

The only restrictions broken are $[2]/[1]C_b$ and $[2]/[0]A_b$.

$\therefore$ L.O. $\notin$ (f.2 ~ g.7)″ and $[2]C_b$ is replaced by $[1]C_b$, to give:*

f.2 ~ *g*.6

As e.9 = e.8 the costs and optimum allocation of Table 9 remain the cost and optimum for (f.2 ~ g.6)″. There are now no broken restrictions, other than $[2]/[0]A_b$.

$\therefore$ L.O. $\in$ (f.2 ~ g.6)″.

The associated total expense is:

$$£(200+130+10+130+50+60)+(40\times 8\tfrac{1}{2})+(13\times 12)+(11\times 5\tfrac{1}{2})$$
$$+(28\times 6)+(11\times 7)=£1{,}381{\cdot}5$$

N.B.

If the total expense found here had been more than £1,542, the amount found for the L.O. in (f.1 ~ g.2).

Then A.O. $\notin$ (f.2 ~ g.6)″ and

A.O. $\notin$ (f.2 ~ g.6) or (f.2 ~ g.7) or (f.2 ~ g.5).

f.2 ~ *g*.8

f.2 ~ *g*.9

* $[2]/[1]C_b$ can be treated as a concave restriction, as $[2]C_b$'s expense variation extended over $[1]C_b$ overestimates the real expense. Vice versa is not true and $[1]/[2]C_b$ cannot be treated as a concave restriction.

On examination it will be seen that both of these groups would have optimum allocations as shown in f.2 ~ g.6. The restrictions broken in fact being such that the choice of group returns to f.2 ~ g.6.

$\therefore$ L.O. $\notin$ (f.2 ~ g.8)″ and (f.2 ~ g.9)″

f.2 ~ *g*.2

f.2 ~ *g*.3

f.2 ~ *g*.4

An argument which disposes of examination of these groups and which could have also been used to dispose of f.2 ~ g.1 is as follows.

Starting with [2]A_a in the choice of groups restricts the choice of costs of factory A to either e.3 or e.4. As for product 1 either of the costs e.3 or e.4 is less than any other at the other factories implies that factory A will always be allocated at least all of product 1 or that part of it that can be handled by factory A. As 51 tons of product 1 are required, this means that factory A will be at full capacity:

50 tons, or be restricted by [1]/[2]A_b to 20 machine weeks.

Now 50 $A_a \sim$ [1]A_b so that concave restriction [1]/[2]A_b will always be broken unless specifically enforced.

Thus, any choice of group containing [2]A_a can be shown not to contain an L.O. unless it also contains [2]A_b.

Hence, the above groups need not be considered further, and neither need f.3 ~ g.1, f.3 ~ g.2, f.3 ~ g.3 and f.3 ~ g.4.

Further, f.3 $\sim$ g.6 and f.3 $\sim$ g.8.

Thus in the remaining three groups f.3 ~ g.5, f.3 ~ g.7 and f.3 ~ g.9, the costs for A and C will remain the same at e.4 ($5\frac{1}{2}$, 6, 7) and e.10 (9, 9, $13\frac{1}{2}$) respectively and will be compared in turn with e.5, e.6 and e.7 at factory B.

f.3 ~ *g*.5

Physical restrictions are applied to factories A and C.

No restrictions of any kind are broken, thus a.L.O. $\in$ f.3 ~ g.5. The total expense associated with it is:

$$£(200+130+10+73+170+80)+(39\times 5\tfrac{1}{2})+(3\times 10)+(9\times 9)+(11\times 7)+(41\times 9) = £1{,}434{\cdot}5$$

f.3 ~ *g*.7

In Table 13 there is more than one optimum allocation. As they all have the same total associated expense, an allocation is *chosen* which just breaks

an artificial concave restriction [2]/[1]B_b) returning the choice of group to a group which has been considered (f.3 ~ g.5).

Thus A.O. $\notin$ f.3 ~ g.7.

f.3 ~ g.9

TABLE 12

Factory \ Product		1	2	3	Physical Restrictions (tons)
A	e.4	[39] 5½	6	[11] 7	50
B	e.5	[3] 10	13	19	
C	e.10	[9] 9	[41] 9	13½	50
Requirements		51	41	11	

TABLE 13

Factory \ Product		1	2	3	Physical Restrictions
A	e.4	[39] 5½	6	[11] 7	50
B	e.6	[12] 8½	10	13	
C	e.10	9	[41] 9	13½	
Requirements		51	41	11	

The allocation given in Table 14 breaks three restrictions. Relaxing [2]/[0]A_b by taking [2]′A_b, the other two restrictions are such to return the allocation to f.2 ~ g.6 and L.O. $\notin$ (f.3 ~ g.9)″.

Local Optima Found. On this first exhaustive examination of the problem three local optima have come to light, not including f.1 ~ g.7.

(1) f.1 ~ g.2 with total expense £1,542.

(2) (f.2 ~ g.6)″ with total expense £1,381·5.

(3) f.3 ~ g.5 with total expense £1,434·5.

However, there need not necessarily be a local optimum in f.2 ~ g.6 or any of the other groups in which [2]′A_b rather than [2]A_b were considered.

Consider these groups. It is certainly possible that the absolute optimum lies within them as (2)'s associated total expense is less than (1) or (3).

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TABLE 14

Factory \ Product	1	2	3	Physical Restrictions
A e.4	[11] $5\frac{1}{2}$	[28] 6	[11] 7	50
B e.7	[40] 8	9	11	40
C e.10	9	[13] 9	$13\frac{1}{2}$	50
Requirements	51	41	11	

From lemma 2 if there is a local optimum within any of these groups it must be when department A_b is at its maximum capacity, as the optimum allocation using [2]′A_b puts the capacity usage of department A_b in [0]A_b.

Therefore, department A_b will be at 40 machine weeks capacity usage. As 40 A_b ~ [1]A_a, department A_a's capacity must be in region 2, and the costs for factory A will be e.4. Unless a modified solution with 40 machine weeks in A_b put more than 15 machine weeks in C_b, f.2 ~ g.7 and f.2 ~ g.9 need not be considered as they have the same costs as f.2 ~ g.6 and f.2 ~ g.8. This leaves four groups to be considered, f.2 ~ g.5, f.2 ~ g.6, f.2 ~ g.8 and f.3 ~ g.9. The associated costs for these groups, of the optimum allocations just applying physical restrictions 50 and 40 tons to A_a and B_a respectively, are found to be £1,404·5, £1,381·5, £1,401·5 and £1,502·5 respectively. Thus A.O. $\notin$ (f.3 ~ g.9).

There are various possible ways by which a solution might at this stage of a problem be found. Obviously applying the simplex technique to each of the remaining three cases to give the optimum allocation subject to all relevant restrictions provides a way of finding the absolute optimum. A better way if the use of the simplex technique is obviously essential is to apply it to the most promising remaining group, i.e. f.2 ~ g.6, to see if the resulting minimum total expense is less than £1,401·5, in which case this would be the absolute optimum. In many cases, especially where the number

of factories concerned is small, it is possible either to obtain the solution by other means, or to find an approximate solution which although it possibly cannot be proved to be the absolute optimum can be shown to be very near it.

One way which sometimes works is to turn the requirements and all other units into the unit in which the requirement is broken. In this case the requirement is [2]/[0] A_b and the limit is thus the "machine week".

Factory A is the cheapest for all products and the costs e.4 are taken as the basic costs, and difference costs from these at the other factories looked on as penalties, i.e. we return to the cost differences of Table 10. Table 10 can be interpreted as showing the penalty (in cost) involved in making an item of a product in any other factory but A. A similar table can be obtained from Table 10 for machine-week cost differences.

TABLE 15

MACHINE WEEK, COST DIFFERENCES W.R.T. FACTORY A

Products / Factory	1	2	3	Physical Restrictions (machine weeks)
A	[25·5] 0	[19·5] 0	0	45
B	6	[21·5] 4	[22] 3	
C	13	6	4½	
Requirements if made at factory A; in machine weeks	25·5	41	22	

This shows the penalty (in cost) of making the items of a product produced in one machine week at factory A at any other factory. It is simply obtained by dividing each item of cost in Table 10 by the number of machine weeks occupied by the item at factory A, i.e. items under products 1, 2 and 3 are divided by ½, 1 and 2 respectively (see Table 2).

If all other restrictions but [2]/[0]A_b are ignored the allocation is made, as shown in Table 15, by requirements in machine weeks *at factory A*. This is the optimum allocation within the restrictions observed. Unfortunately*

* Otherwise a solution could be found easily by this means.

it breaks [2]/[0]A_a and all that can be shown from this is that factory A, by lemma 2, will also be maximum capacity in department "a", i.e. 50 tons.

$f.2 \sim g.6$

It can be seen that factory B will be at maximum tonnage so Table 11 can be used. It is wished to manipulate the 50 items allocated to factory A to reduce the machine weeks occupied there from $55\frac{1}{2}$ to 45.

TABLE 16

Product / Factory	1	2	3	Physical Restrictions (tons/machine weeks)
A	[11] 0	[28] 0	[11] 0	50/45
B	[40] 0	1	3	40
C	$\frac{1}{2}$	[13] 0	$3\frac{1}{2}$	
Requirements (tons)	51	41	11	

If 1 unit of product 3 is transferred from A to B and 1 unit of product 1 is transferred from B to A:

there is an increase in cost of £3 for a reduction of $1\frac{1}{2}$ machine weeks at A. i.e. penalty of £2 per machine week.

By considering the various possible transfers of this nature it is seen that the above transfer and the one similarly transferring items 1 and 2 between A and B both have the lowest penalty of £2 per machine week. The optimum allocation then is $10\frac{1}{2} \times £2 = £21$ dearer than the one for (f.2 ~ g.6)′. As none of the possible optima to f.2 ~ g.6 break any restrictions they are local optima. As the total expense is now still only £1,403 compared, it is admitted, to £1,401·5 in (f.2 ~ g.8)″ which breaks [2]/[0]A_b, it is obvious that the absolute optimum is in fact contained in f.2 ~ g.6.

$$A.O. \in f.2 \sim g.6$$

The Absolute Optimum. The absolute optimum allocation can be anywhere between the allocations on the opposite page, provided the total

tonnage at each factory is kept the same and 45 machine weeks are occupied at A.

(32, 7, 11) tons at A
(19, 21, 0) tons at B
(0, 13, 0) tons at C

and

(18, 28, 4) tons at A
(33, 0, 7) tons at B
(0, 13, 0) tons at C

They have associated total expense of £1,403.

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Wege und Irrwege mathematischen Denkens in Wirtschaftstheorie und Unternehmenspolitik.

W. BIERFELDER.

Marktwirtschaft und Verbrauch 8 (*Gesellschaft für Konsumforschung*), 1958. 109 *pp.*

Bierfelder discusses the proper place of mathematical reasoning in the "Wissenschaften", a term used here to cover not only the sciences but all organized bodies of knowledge (e.g. history, economics, philosophy). The Wissenschaften are graded, following Wenzel, in four levels: inorganic, organic, mental and spiritual. As one progresses from level to level, the entities dealt with become more and more affected by individual differences, and therefore less and less properly handled as something amenable to mathematical treatment. Bierfelder admits that trying to put the higher levels into mathematical form may provide suggestive insights and may form a convenient way of exhibiting the logical connexions of a subject, but attacks the use of mathematical methods at the higher levels to "prove" anything. At the most, it may suggest results that must be confirmed by observation. Prediction must give way to well-reasoned foresight. He attacks mathematical methods of handling business decisions (specifically, linear programming), but appears to have an unduly mechanistic idea of the sorts of results that come from such analyses. The book is filled to excess with quotations from and references to other workers.

T. E. EASTERFIELD

Planning of Experiments.

D. R. Cox.

Chapman and Hall, London; John Wiley and Sons, New York, 1958. vi + 308 *pp.* 60*s.*

It would be interesting to know, but difficult to ascertain, just what mathematics and statistics are used by operational research people. In particular, how often is an experiment designed and analysed along classical lines? The guess of the reviewer is, not very often. Operational research is a very different field from research and development where

control and choice of variables are more readily achieved. Operational research is more concerned with constructing reasonable models of existing situations and finding out what would follow if certain underlying assumptions were altered. Statistics enter into this model-making in general, but more often in its data-analysis, rather than in its experimental, rôle.

However, experimentation to investigate technological factors will be necessary in certain problems. We must also remember the extensive use of simulation techniques to study models of complex problems. These exercises, with the use of electronic computers, are capable of studying the effects of a number of factors. They are, in a sense, experiments, and whoever employs this approach is more likely to do so intelligently if he has a basic knowledge of statistical experimental design. We conclude that for either of these purposes an operational research library must include reference books on this topic.

There are in existence several textbooks on this subject, some of them very good. To a greater or lesser extent, they treat the subject as an application of the analysis of variance. This is admittedly a proper tool to use, but one which requires a fair level of statistical knowledge to use with understanding. Dr. Cox has adopted a different approach. In his preface he says: "The book is an account of the ideas underlying modern work on statistical aspects of experimental design. I have tried, so far as is possible, to avoid statistical and mathematical technicalities, and to concentrate on a treatment that will be intuitively acceptable to the experimental worker, for whom it is primarily intended." He has admirably fulfilled this task.

The book, of 300-odd pages, has fourteen chapters of which the first nine give essential ideas and the more basic designs such as the randomized block, Latin square, simple factorial; the remaining chapters deal with more advanced topics, such as incomplete block designs, fractional replication, cross-over and trend-free designs. There are extensive references to papers by chapter, a short general bibliography, and in the Appendix, tables of random permutations and random digits.

The whole emphasis of the book is on the experimental situation being studied. This is discussed, illustrated with examples drawn from a wide variety of sources, and it is shown how the statistical treatment follows logically from practical considerations. How the essential calculations are carried out is shown to a large extent without resort to symbolism. The style in which the book is written is clear and direct, although, as the

author points out, some parts are quite closely argued and are not necessarily easy reading.

Certain topics are dealt with in a more satisfactory manner than in other books. There is a chapter on randomization, in which, for example, the matter of rejection of an arrangement produced by randomization for a small experiment is discussed. This is something that often perplexes an experimenter who is trying to use his statistics conscientiously. Again, Chapter 6, on basic ideas about factorial experiments, is an excellent exposition. The types of factors that may occur are fully discussed, it being shown in what cases the response surface model is appropriate, and in what cases it is "main effects and interactions". What often seems a muddling subject is here presented in a clear, concise and helpful way.

This book should be welcomed by all who are concerned in any degree with experimentation. Those who are new to the subject, whether mathematicians or otherwise, would be well advised to read this book first. Those who do already know something about the subject will certainly benefit, and the statistically-minded experimenter should find many of his difficulties resolved. Further, the book should certainly find a place in every operational research library.

N. JESSOP

Linear Programming.
SAUL I. GASS.
McGraw-Hill, New York, 1958. xii + 223 *pp.* 60*s.* 6*d.*

Linear Programming.
ROBERT O. FERGUSON and LAUREN F. SARGENT.
McGraw-Hill, New York, 1958. xiv + 342 *pp.* 77*s.* 6*d.*

Two books from the same publisher, with the same title, but quite a difference between them. Gass's book is a proper textbook, based on a course of lectures given at the Graduate School of the U.S. Department of Agriculture.

Ferguson's and Sargent's effort belongs to the category of "how to do it yourself" books. In saying this, no slight is intended. In fact, if this type of book has to be published, the work under review is one of the best. It is only that this reviewer, at least, finds difficulty in imagining the potential readership for such books. According to the authors, it is intended "primarily, but not exclusively, for people engaged in management activities at all levels in the firm . . .". Surely the higher level manager,

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who wants to learn about linear programming, has hardly the time to work through a book of some 350 pages, setting out computational procedures in great detail, whilst the people at lower levels of management, who might want to become proficient in this field, will need somewhat more of the theory behind the computations and case histories than is given here. To a certain extent the authors themselves recognize this, stating in their last chapter, headed "Putting linear programming to work", that "the introduction of linear programming into a business should be less difficult, if the organization has already a functioning operational research group . . .".

In keeping with the idea of making matters as easy as possible, the authors show a definite bias against the simplex method, going as far as to state: "It (the simplex method) is prone to error, except when solved by an electronic computer."(!) In consequence, section II of the book, which could be subtitled "How to compute linear programmes", devotes a chapter each to the modi—ratio—analysis—and index method. Nevertheless, the two chapters on the simplex method are written in a clear and simple style and could well be used as a basis for training computing staff.

Section III, on applications, is a collection of fairly simple case studies. Again, the explanations and layout of tables are clear and easily understood, and this part could be quite useful for an introductory course in linear programming if given in conjunction with some basic theory. True there is an appendix called "A mathematical description of the simplex method of linear programming", but this is too short and difficult for readers without the requisite mathematical background, and rather small meat for those who have it.

The book also has a rather scanty bibliography and, for British conditions, a somewhat high price.

There is no need to say much about Gass's book except that it is a very good textbook, for graduates, or undergraduates in their last year, and that it requires mathematical knowledge of a fairly high standard. There is an introductory chapter on matrices, vectors, convex sets, etc. which can serve as a refresher for those coming to linear programming some time after their university.

The next five chapters are devoted to the mathematics of linear programming and the simplex method.

Two further chapters deal with parametric linear programming and additional computational techniques respectively. The latter chapter also contains a list of available computer programmes for solving linear

programming problems. Unfortunately, the only British machine mentioned is the Ferranti Pegasus.

The collective heading of the last three chapters, "Applications", is a misnomer, as one of them deals with the transportation simplex, and one with the relationship between linear programming and the theory of games.

The book ends with an extensive bibliography of linear programming applications.

It can thoroughly be recommended as a textbook or reference work, and if one has to make a criticism, it is that in a really comprehensive reference work on linear programming one would like to see additional chapters on linear programming under "Uncertainty" and on the Ford–Fulkerson method of solving the transportation problem.

E. KAY

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Productivity Measurement in Great Britain—A Survey of Recent Work
T. E. EASTERFIELD.
H.M. Stationery Office, London (for D.S.I.R.), 1959. 79 *pp. gratis.*

The survey discusses 109 main references and in the text also makes comment on several other relevant papers. There is sufficient information to make the booklet very readable and the author has not been afraid to be critical. Indeed, it is interesting to see the personality of the author emerging so clearly in a publication from a government department.

The main headings are:

I. Studies of multiple inputs.
II. Studies emphasizing the problem of inhomogeneous output.
III. Factors that may affect productivity.

The methodology involved in most of the references is discussed by Dr. Easterfield and as such would be of value to many operational research workers.

H. G. JONES

Handbook of Automation, Computation and Control (Volume 1).
EUGENE M. GRABBE, SIMON RAMO and DEAN E. WOOLDRIDGE (Editors).
John Wiley, New York; Chapman and Hall, London, 1959. xx + 974 + 20 (index) *pp.* 136*s.*

There are a number of modern professions, based on mathematics—operational research is just one of them—in which one needs to have a knowledge of the subject matter of a number of related scientific disciplines.

A handbook collating the material of these various disciplines is therefore to be welcome. To quote Professor Gordon S. Brown, of MIT, who wrote the foreword to this volume, ". . . handbooks are an essential component of the technical literature, when they provide the unity and continuity that are requisite".

It is the intention of the editors to provide "in one place the available theory and information on general mathematics, feedback control, computers, data processing and systems design". To judge by this first volume (altogether there will be three), they have succeeded admirably.

The volume is divided into five parts: General mathematics, Numerical analysis, Operations research, Information theory and transmission, and Feedback control. To give an idea of the real scope of this book one can do no better than quote the chapter headings:

Part (i): Sets and relations, Algebraic equations, Matrix theory, Finite difference equations, Differential equations, Integral equations, Complex variables, Operational mathematics, Laplace transforms, Conformal mapping, Boolean algebra, Probability, Statistics.

Part (ii): Numerical analysis.

Part (iii): Operations research.

Part (iv): Information theory, Smoothing and filtering, Data transmission.

Part (v): Methodology of feedback control, Fundamentals of systems analysis, Stability, Relation between transient and frequency response, Feedback system compensation, Noise random inputs and extraneous signals, Non-linear systems, Sampled data systems and Periodic controllers.

As each chapter is written by a different author, variations in quality and level are unavoidable. All of them, however, fulfil their purpose, although some are fairly hard going. Each chapter ends with a small but adequate list of references. Parenthetically, the chapter on operational research, written by E. L. Arnoff, is a very good digest of Churchman, Ackoff and Arnoff, "Introduction to Operations Research", and also has the most extensive bibliography.

Apart from presenting its subject matter in a concise form, a useful handbook must have three further attributes: freedom from mistakes or misprints, a layout facilitating easy reference and readable print. On this score also, one must give the editors full marks.

It is rather difficult for a single reviewer to check a book of nearly a thousand pages, abounding in formulae, thoroughly; however, in the scrutiny I carried out I could discover only one mistake, probably a

misprint, on page 13-16, where the statement on the degrees of freedom of the χ^2-distribution should read $k-1$.

Pages are numbered by chapters only, so that 13-16 is page 16 of chapter 13. The index is very complete and reference by means of this page-numbering easy. Printing is of the highest standard. Small print is avoided throughout except for the reference list at the end of a chapter.

Volume 2 will deal with computers and data processing, and Volume 3 with systems and components.

Despite its price, this is a book which is highly recommended.

E. KAY

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Mathematik, Technik, Wirtschaft (MTW), Volume 6, Nos. 1 and 2.
(*A quarterly for modern methods of calculation and automation.*)
Stiasny Verlag G.m.b.H. Graz, Annenstrasse 65, *Austria*, 1959. *Austrian s.* 20*; s.* 75 *yearly.*

The "MTW Mitteilungen", published in German, have appeared since 1954 as the technical communications of the Mathematical Laboratory at the Polytechnic in Vienna. The quarterly now appears in a new edition continuing on this authoritative basis, aimed to serve a wider public in management, industry and commerce. Consequently, it contains articles of a technical nature on computing techniques, systems theory and operational research, but more emphasis is placed on applications of data processing and automation, and discussions of practical considerations with case studies.

The wide coverage of the quarterly may be seen in the articles appearing in the first two copies received: data processing; fundamentals of input–output analysis; the nature and limitations of automata; conformal presentation on analogue computers; the use of data processing equipment in the office; electronic computers in the U.S. satellite programme; punched-card system of the Austrian land-registry office; progress on the Austrian reactor. The coverage would seem to be very ambitious were it not for solid support from commentaries, reports on conferences held in many countries in Europe, programmes of societies, news from manufacturers, book reviews and abstracts. The reviewer is puzzled, however, by the inclusion of a few abstracts dated 1957.

The quarterly presents a broad picture of the "state of the art", on the Continent in particular, and is for this reason likely to gain considerably in importance in the near future.

W. L. VAN WARMELO

The Exponential Curve of Science.
DEREK J. PRICE.
Discovery, **17** (6) (*June* 1956).

The future of science is vitally important yet little attempt has been made to study it scientifically: is it possible?

Examination is here made of the past and present rate of growth of science. Available statistics variously relating to scientific manpower and output show that since 1700 the growth of science has been exponential and has doubled in size every 10–15 years. This rate far outstrips the rate of growth of other human activities.

Experience of exponential growth has shown that the actual growth curve is modified to an S-shape as the original exponential curve approaches saturation level: taking the growth curve of scientific manpower and assuming a reasonable saturation point, the indications are that a critical level will be reached in some 20–30 years and the consequent retardation in growth rate is bound to have serious consequences. Possibly the current shortage of scientific manpower confirms that this situation already applies.

In addition to this "external" repressive influence of saturation on the growth of science, two "internal" influences may be working to inhibit progress:

(1) The growth of scientific literature has also increased exponentially since 1700 and appears to have reached a critical level; further work is required on techniques of communications.

(2) Measures relating the size and achievements of science suggest the law that the size of science increases approximately as the cube of its achievements; this is related to the problems both of specialization and of communication.

The work described here is crude and approximate only, but it suggests methods capable of refinement and extension which need to be employed if the predicted crisis in science is to be met.

The Progress of Science: The Future and the Strategy of Science.
Discovery, **19** (1) (*January* 1958, *Editorial Article*).

The creation of a better future rests upon science. In the last 200 years science has roughly doubled in size every 10 years: this growth has been largely haphazard and contingent on financial resources, but it appears likely to be halted in the near future because of manpower shortage.

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The demands on science, however, will continue to expand. In this situation, science must be prepared to apply scientific methods to itself and to devise a strategy of science aimed at producing the maximum effort from available resources. This situation is familiar enough to scientists concerned with applying scientific method to production difficulties in other fields.

The strategy of science must be distinguished from its tactics. By strategy is meant largely the continued and open-minded revision of scientific values and standards, the allocation of priorities for research topics, the development of measures of efficiency and a readiness by science to observe itself dispassionately and to submit its conclusions reached thereby to scientific verification. Finally, science must be prepared to deal with the problem of both internal and external communication.

International Scientific Meetings and Exhibitions: Scientific Information Conference.

Discovery, **20** (2) (*February* 1959, *Editorial Article*).

The problem of the storage and retrieval of scientific information is currently receiving much attention from scientists and librarians. An International Conference on Scientific Information, dealing with these topics, was held last year in Washington, D.C.

Sir Lindor Brown queried the need for the quantity of scientific literature now produced and suggested that more qualitative distinctions be made. The reading capabilities and habits of scientists were discussed and the need for research on these subjects was stressed. The rival merits of abstract and abstract/index systems were considered.

The need for more reviews was raised and some discussion was held on the best form these should take. In discussion on computers and "information language" the hope was expressed for the general development of mathematical retrieval theory. The role of scientific societies in disseminating information and the possibility of financial support and international co-operation were also discussed; no agreement was reached on "international information centres".

Several items of equipment were exhibited, including translation and character recognition machines, punched-card apparatus, micro-film equipment and a high-speed analogue to digital converter.

No recommendations or resolutions were made by the Conference, but the general feeling was that too much information is produced and that some selective procedure is necessary.

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The Science of Science: What Scientists read and why.
CHRISTOPHER SCOTT.
Discovery, **20** (3) (*March* 1959).

The problem of communication between scientists, posed by the increasing output of scientific literature, is currently receiving a great deal of attention. Two possible solutions are to reorganize:

(1) the structure of scientific research;
(2) the information services.

Research so far on this problem has concentrated on the latter and the former has received little systematic attention. From this work can be seen emerging the foundations of a "science of science"; that is, a scientific study of scientists themselves.

The results are summarized of a 1956 Government social survey enquiry in which 1,082 industrial research personnel, mainly technologists, were interviewed on their scientific literature reading habits. It was found that the amount of technical reading was small and that libraries were little used and rarely criticized: much technical reading was done at home and was indulged in from general interest and to gain ideas rather than to answer specific questions.

The overall results suggest the following hypothesis: "The technologist reads technical literature mainly for interest or news; only rarely does he consult it to find a specific piece of information."

More research is needed on the "scientific reader", but the above hypothesis contradicts the generally accepted belief of librarians that their function is to provide an information service and has consequences for their efforts towards improvement. The main problem is how to present scientific information attractively enough to capture the attention of the relatively casual reader.

International Journal of Abstracts: Statistical Theory and Method.
Oliver and Boyd, London. Annual subscription £5.

The aim of this new journal is to give complete coverage in the field of statistical theory and new contributions to statistical method as published after 1 October, 1958.

Abstracts will be in the English language and will be of up to 400 to 500 words, the format and binding being such as to permit them to be split and filed in page or card form. A new scheme of classification has been developed and pages are colour-coded in accordance with the main sections of the classification.

The General Editor is Dr. W. R. Buckland, and international teams of regional editors provide a world-wide representation.

The journal will appear quarterly and contain approximately 1,000 abstracts per year.

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News and Notes

Swedish Operations Research Association

We are advised that the Swedish Operations Research Association was founded on 26 February, 1959.

The Council elected for 1959 is composed as follows:

President: Professor Harald Cramér.

Vice-President: Professor Lamek Hulthén.

Secretary and Treasurer: Docent Tore Dalenius.

Other Officers: Professor Tryggve P. Frenckner and Fil. lic. Jan-Robert Eklind.

The address of the association is:

c/o Institute of Statistics, 70 Rådmansgatan, Stockholm Va, Sweden.

Soviet Census of Industrial Resources

In a despatch from Moscow, Harrison Salisbury (*New York Times*, 24 May, 1959) refers to the comprehensive census of industrial resources to be undertaken in the U.S.S.R.—the first there since 1925. The correspondent foresees the use of these census statistics for a gigantic reassessment of the Soviet economy and for linear programming and input–output analyses, directed particularly to problems regarding the development and location of new industries, for which sufficient electronic computer capacity is said to be available.

Wassily Leontief and other American economists are co-operating in this enterprise which can be expected to lead to major developments, not only in the Soviet economy, but also in the matrix techniques themselves.

First British Computer Society Conference Cambridge, June 1959

To an operational research worker, the electronic computer is considered either as a tool for research or as part of a control system. At the first British Computer Society conference, held in Cambridge from 22 to 25 June, 1959, it was not perhaps surprising that the emphasis lay almost

exclusively on the direct uses of computers for scientific, engineering and operational research and for routine commercial calculations.

The papers of direct operational research interest were those by Mr. J. Harling of Urwick, Orr & Partners on inventory control and by Mr. G. S. Galer of Shell Chemical Co. on linear programming.

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Mr. Galer especially brought home the need for computers in the economic life of a large company and the requirement for larger and faster computers than are available at present (this topic in fact occupied part of the conference programme). The Royal Dutch/Shell Chemical Co. carry out allocations of their manufacturing capacity for a number of petroleum chemical plants each producing a wide variety of products. At present the plans are formulated every 3 months looking about 12–18 months ahead. When the larger machines already available are used it will be possible to look ahead several years.

Mr. Harling in covering some of the theoretical aspects of inventory control revealed how little sophisticated but also practicable work had been done in this field on computers and how great was the scope for further developments.

The selection of programmers came under discussion and, as usual, supporters and opponents of special aptitude tests were found to be about equally divided; no general conclusions were reached on the type of person most likely to be successful. The advantages were stressed of the increasing availability of automatic coding systems whereby many more people, operational research workers included, would find they could use computers easily themselves.

Although it was refreshing to attend a conference where "Monte Carlo's" were not mentioned, it did suggest the possibility of investigating the effects of the introduction of computers into a company's activities, e.g. production control by means of a computer simulation. Perhaps, in this way, some of the so-called intangibles could be resolved and assessed.

R. G. MASSEY

Operations Research Society of America
Sixteenth National Meeting

THE Sixteenth National Meeting of the Operations Research Society of America is to be held in Pasadena, California, at the Huntington-Sheraton Hotel on 11, 12 and 13 November 1959.

Technical sessions will be held on:

Missiles and space systems
Military strategy and tactics
National security objectives
Traffic and transportation
Evaluation through experimentation
Business management and planning
Production planning
Medical applications
Simulation of operations
Mathematical models
Education for operations research
Reliability
Military and commercial aircraft operations.

In addition, panel sessions have been organized on:

Production planning and scheduling
The future of operations research
Effective presentation of operations research material
Military air transportation
Information retrieval.

Non-members are cordially invited; registration can be made at the meeting. Company or special tables may be reserved for the banquet and luncheon events. Hotel arrangements should be made directly with the hotel.

Second West Coast ORSA–TIMS Meeting

THE second West Coast ORSA–TIMS meeting will be held at the U.S. Naval Postgraduate School, Monterey, California, on 7 and 8 April 1960. The meeting is sponsored jointly by the Northern and Southern California Chapters of TIMS and the Western Section of ORSA. There will be both invited and contributed sessions of theoretical and applied character. Abstracts of contributed papers must reach the Programme Chairman by 1 February 1960. The committee chairmen for this meeting are:

General Chairman: Prof. D. Teichroew, Stanford University.

Arrangements Chairman: Prof. T. Oberbeck, U.S. Naval Postgraduate School.

Programme Chairman: E. Koenigsberg: Touche, Niven, Bailey & Smart, 100 Bush Street, San Francisco 4, California.

Second International Conference on Operational Research

THE International Federation of Operational Research Societies announces that the Second International Conference on Operational Research will be held from 5 to 9 September 1960 at Aix-en-Provence, France, by arrangement with Société Française de Recherche Opérationelle, 27 Rue Laffitte, Paris 9.

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The conference is intended primarily for members of the Societies belonging to the International Federation, but anyone working in the field of operational research, or having a direct interest in it, will be welcome to attend if places are available: the number of delegates attending will be limited.

The Second Conference will develop some of the topics already covered at the first conference held at Oxford in 1957, and will include papers on most of the following subjects:

Computers and operational research
Mechanism of decision making by groups
Human factors
Operational gaming
Some new areas of mathematical programming
Evaluation of inventory control
Measures of organizational effectiveness
The control of production
Sequential procedures
The theory of graphs and its application
Catering for catastrophes
Operational research at the strategic level in industry
The operational research method.

Application forms and full information will be obtainable from the Secretaries of member Societies.

News of Members

New Full Members

Dr. L. F. Broadway—Head of Research Laboratories, Electrical & Musical Industries Ltd.

Dr. F. W. Dresch—Manager of Industrial Research Unit, Stanford Research Institute, U.S.A.

Mr. H. Kerner—Manager O.R. Department, Associated Engineering Ltd.

Mr. F. D. Robinson—O.R. Department, Glacier Metal Co., Ltd.

Mr. G. Thakray—O.R. Unit, Distillers Company.

Mr. R. A. Ward—The Royal Institute of Public Administration.

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Changes of Appointments

Dr. A. H. Warner—Previously with the American Operations Research Group in London has returned to the United States.

Mr. R. R. P. Jackson—Previously with Shell Mex & B.P., is now with B.I.S.R.A.

Mr. W. D. Ray—Previously with B.C.U.R.A. now with Laporte Industries Ltd.

Dr. S. K. Shear—Previously Joint U.S. Operations Research Groups Liaison Representative in London, now with Operations Evaluation Groups, Washington 25, D.C.

Dr. A. J. Skey—Previously with Canadian Joint Staff, London, is now with Amoco Chemicals Corpn., London.

Dr. G. Gregory—Previously with the University of Birmingham, is now with the University of Melbourne.

Mr. H. G. Jones—Previously with the U.K. Atomic Energy Authority, is now with Richard Thomas & Baldwins Ltd., Newport.

Obituary

It is with great regret that we have heard of the sudden death of Jack Worrall of Shell Mex & B.P. on 12 September.

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A Survey of Operational Research in British Industry*

B. H. P. RIVETT
National Coal Board

This paper surveys the recent development and present position of operational research in British industry. It makes a critical examination of the relationship between the techniques available and the types of problem on which they are deployed; it surveys the type of people now carrying out operational research and comments on career opportunities. The paper also considers the integration of operational research within industrial organizations and suggests ways in which it can be related to other management services.

1. Introduction

THE task of this paper is to outline the current development in industrial operational research, to sketch our strong points and to indicate the areas in which the subject is weak. It is hoped that this will supplement earlier papers by Goodeve and Ridley[1] and Gander[2].

It will be wise at the beginning of the paper to make two important reservations. In the first place the paper has to be subjective in character. It is possible to find out something of the problems on which our people are currently engaged. The extent to which it is possible to classify these problems is open to debate and in any event the classification itself is subject to the present fancy of the particular scientist concerned. The second reservation stems from the first and is that the views expressed herein are those of the writer and do not necessarily represent the official opinion of the Operational Research Society.

Two approaches have been followed to get the basic information for this survey. The first was a survey of a random sample of the membership (full and associate) of the Operational Research Society. One hundred questionnaires were distributed of which seventy-six were returned. The

* Paper presented at the Annual Conference of the Operational Research Society at Harrogate on 10 June, 1959.

second was to obtain from a number of senior operational research workers, answers to specific questions; this gave information on a further 180 workers.

It is well to remember in surveying operational research that, so far as other sciences are concerned, it is still quite young. Although no one would claim that no operational research had been done prior to the coining of the phrase in 1936, it is only in the last 20 years or so that the subject has been widely studied and developed. It should not be expected therefore that the subject should stand at present as a coherent whole and that the techniques of the subject should be applicable to all industrial subjects. Operational research is still in the stage of fast development and, in common with any other science, will only continue to develop as long as it is being posed problems which at the moment of their posing are unsolvable.

The development of any science is conditioned primarily by the interaction of the scientists concerned and of the subject matter. This paper will therefore begin with a short survey of the people who are currently doing operational research, their background, their education and their salaries. The following sections will then consider the types of work on which these people are engaged and the techniques being used. The concluding sections will deal with the organization of the operational research groups and the consideration of how effective operational research groups have been.

2. *Operational Research Workers*

In this section data will be given regarding the age, experience and qualifications of operational research workers. Apart from one or two special cases no information will be presented to give similar data for scientists generally; this is simply because of the difficulty of obtaining such data.

The first point to note is the comparative youth of the scientists in our groups. The average age is 29, but many are in their early twenties. The average experience in operational research is 4 years. The average time that people have been in their present jobs is 3 years, but 75 per cent have been in their present jobs less than 3 years; this does not necessarily reflect high turnover of staff, since the operational research field has expanded greatly in the past 10 years.

The essential feature of operational research is that it is *research*, and operational research workers should as research workers rank in the

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front line. Although it is dangerous to place too much emphasis on first degree qualifications, it is disappointing that of the people surveyed only 12 per cent had first class honours. In addition, only 12 per cent had done post-graduate research leading to a degree. There is a predominance of mathematicians and statisticians in our operational research groups, but it is particularly interesting to note the wide range of subjects in which operational research workers have specialized, including meteorology, astronomy and logic. Some senior operational research workers are disturbed at the high proportion of mathematicians and statisticians in operational research. During these early stages of the subject, however, the degree speciality of a person may not matter much, so long as he eventually becomes an operational research scientist. A large number of people surveyed had gone to full-time operational research courses, mostly of the two-week variety. Two operational research groups had so far put on their own internal course in operational research to train their recruits; these were the groups at the United Steel Companies and at the National Coal Board.

At present the literature on operational research is not overwhelming in its extent. Most people seem to have read some or all of Churchman, Ackoff and Arnoff, and some or all of the *Operational Research Quarterly*. The American journals, *Operations Research* and *Management Science*, were also read extensively. Operational research workers do not seem to spend much of their own money in buying books on their subject, although this may be a common trait of scientists generally.

The overlapping of other scientific societies was not as great as might have been expected. The only society with which the Operational Research Society has a significant common membership is the Royal Statistical Society, to which one-third of the people surveyed belonged.

About half the people who were approached wanted to stay in operational research until they retire. Of those who wished to leave the subject the majority wanted to go into some form of administration or management (some comments on the extent to which this desire is likely to be satisfied are given in Section 7).

Most people felt they are paid roughly the same as similar specialists in their firm who are not doing operational research. The following table is based on a general impression gained in discussing salaries with senior operational research workers and gives an average salary for a competent operational research worker or a competent scientist recruit to operational research.

It is interesting to compare these figures with a survey carried out recently amongst the Fellows of the Royal Statistical Society.

TABLE 1

Experience in Operational Research (years)	Age (years)		
	20	25	30
	Salary		
	£	£	£
0	700	900	1,200
5	x	1,100	1,650
10	x	x	1,900

TABLE 2

Age	Median Salary
	£
25–29	870
30–39	1,274
40–49	1,805
50–59	over 2,000
60–69	over 2,000

The operational research salaries provide a stimulating incentive to the young graduate coming fresh into the subject, but it is open to question how far these men can be retained into their years of maturity (see Section 7).

3. The Type of Work Now Being Tackled

Current operational research literature shows a basic pattern of work in which the dominant note is mathematics or statistics. The main emphasis in published work is on the manipulation of models and the derivation of hypotheses. It is rare indeed to see the verification of the model attempted and to this extent the subject, *in so far as the published papers are concerned,* is not a science but sophisticated crystal-gazing. Industrial operational research workers know that the literature does not reflect the activity, but it is not surprising that some scientists outside operational research look on the subject as a branch of applied mathematics or statistics. It is however a pity, indeed it is a serious criticism, that the *Operational Research Quarterly* is not bulging with completed case studies which are the subject of comment and criticism. A scientific journal

is the first line of communication among scientists and until a new scientific work is exposed in the literature it cannot be regarded as a serious contribution to the subject. Operational research workers generally share a responsibility for the quality and quantity of papers appearing in their journal.

The task of an industry can be divided very broadly under the following headings:

Purchasing	Finance
Stock holding	Personnel
Production	Marketing
Research and Development	

The ultimate stage in the development of operational research in industry will be achieved when operational research work is carried out at a level above all these separate areas, which will involve studying the general strategy of an industry and resolving the conflicts which are inherent and inevitable in the compartmentalization. Some operational research groups are carrying out studies in this strategic field, but so far not very much effort has been deployed here. Before moving into this field it is necessary to carry out basic studies in the separate parts of an organization; sub-optimization must precede optimization. A second reason for not much work being done in this strategic field is that there is a shortage of mature, able operational research workers who can move at will within an industry without giving offence, who have the intrinsic scientific ability to carry out strategic studies and who also have the personality to encourage a board of directors to repose this confidence in them. Many operational research workers are frustrated in that they are not yet working in this strategic area. They feel like a doctor who is restricted to examining the patient's feet, while all the time his real trouble is somewhere else. It would be wise, however, not to underestimate the effect of the shortage of mature operational research workers and anyone who has been asked to advise a firm on their first operational research appointment meets two difficulties:

(a) to persuade the firm to appoint a first-rate man and not simply an operational research technician, and

(b) to find such a man.

Within the component parts of the organization, which is where most operational research groups work at present, the field of activity is still unbalanced. Little work is being done in the area of personnel management—human relations, absenteeism, accidents, morale, incentives,

selection and training of staff and so on. The obvious difficulty here is that of getting the means of measurement; until the scientist can measure, he is impotent. In this particular area we largely depend on the social scientist to produce the measures. The importance of human factors on the economy of the firm should not be underestimated and more social scientists should be recruited to operational research groups, provided they are real scientists specializing in the human problems area.

Another area in which little activity is in progress is that of purchasing. The nature of purchasing, with the twin requirements of market forecasting and of competition with other purchasers, would seem to make the field a fertile one for the scientific approach. One inhibiting factor would probably be the speed with which results are required; this point was well brought out by Crack in a paper at the 1958 Harrogate Conference.[4]

The third field in which not much work is currently in hand is that of marketing. Market research has, of course, been widely developed in recent years, but so far there has been little liaison between market research and operational research. There is an obvious link between market research and the social scientist in the field of motivation research. It would be interesting to see how far these new techniques can produce methods of measurement which can be used by the operational research workers. As a particular part of the marketing field there is advertising, in which the mystique of the craft has so far almost completely excluded the scientist. To derive and confirm models for consumer response to advertising is extremely difficult; this should, however, be a valuable area for operational research in view of the amount of money being spent by industry on advertising. Allied with this is the problem of fixing the selling price of a commodity; the derivation of the true cost of producing a particular line and hence the optimum selling price is an obviously important topic on which much work remains to be done.

What then are the main problem areas in which operational research work is currently being carried out? As might be expected the main fields are finance, inventory, production and transportation. In finance the main effort is in some form of cost accountancy and cost control; the important aspect of financial control is not receiving the attention it deserves. The difficulty of getting from the cost accountants basic data which operational research workers feel are "true" is one of the inhibiting factors here. The cost accountant is in a key position in industry since it is he who produces the basic cost information.

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The operational research worker and the cost accountant must work together and more effort should be deployed by both sides in getting to know the other's speciality.

Inventory control is currently the subject of much activity and the following section will comment on the adequacy of the theory of inventory control. Operational research workers, however, have done a great deal of useful work in this area. In a similar way transportation studies have been tackled quite widely. Unlike inventory control, the techniques seem well adapted to the real life problem.

The manifold, interlocking and competing demands in production make this a fertile field for useful operational research activity. The operational research specialist has been accepted here and this is probably the most common area of work.

4. Techniques of Operational Research

In any operational research project the first stage is to set out the objectives which have to be satisfied by the system. Reality is then expressed in terms of a manipulable language (usually mathematical) and manipulations are carried out according to basic rules to derive, in terms of a meta language, an "optimum"; this optimum is then translated from the meta language into terms of real life and the real life optimum is pursued and checked to see whether the whole analysis is justified.

Some operational research workers regard the subject as a collection of mathematical or statistical tools, such as queueing theory and linear programming. On the other hand there are those who seek to formulate operational research as a mystique comparable to the mystiques which operational research workers deplore in other specialities and other management functions. Operational research is neither of these two extremes. It is a whole, complete study in which mathematical techniques bear the same relationship to the study as the rules of grammar do to literature.

The most important part of an operational research job is the formulation of the problem and its primary exposition in the meta language. Of equal importance is the final act of translation from the meta language into reality. The actual manipulation methods are of secondary importance only and indeed in many studies this stage is comparatively trivial. It is unfortunate that both the American and the British operational research journals do not make this point clear in the papers presented therein.

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The emphasis and interest of the straightforward mathematical and statistical methods vary from year to year. Some time ago linear programming had an exotic aura. All were convinced of its usefulness; if only problems could be found which it would solve. Now, however, it is widely applied to transportation problems and to problems of the allocation of capital resources. The ubiquity of computers, and the ease with which linear programming can be programmed into a standard form on them, has meant that linear programming without tears has become an everyday event. At the same time it is being realized that the use of this tool opens up a field of research in that serious basic restrictions are exposed. The first is an obvious one: linearity. Industrial life is often non-linear. Second is the problem of the imprecision of the parameters put into the basic equations and inequalities. If these parameters are subject to error what is the effect on the solution both in terms of the subject of optimal policy and the distribution of expected pay-off? We may paraphrase Box and term this the "robustness" of the solution.

Queueing has been widely developed as a pure statistical theory, largely separate from industrial needs. Although some industrial queueing situations can be expressed in terms of the standard queueing theory models, many of the distributions have not heard of D. G. Kendall. Other complications arise when an algebraic solution of the queueing model is attempted. At this stage resort is often made to simulation and Monte Carlo methods: these have been greatly developed in recent years and are obviously attractive. One of the virtues of a simulation approach is that it forces the operational research scientist to express with clarity the fundamental patterns of behaviour of the process that he is simulating. At the present time one is probably seeing only the beginnings of the revolution that computers are going to cause in our scientific methodology in general and in mathematical techniques in particular.

Some of the queueing techniques are the models and methodology for inventory control; this is a field in which much work has been done in the derivation of models and methods. It is, however, rare to find an inventory situation which can be solved by classical methods.

By far the most common techniques at present used are ordinary statistical and mathematical methods. Statistically designed experiments are rare (all would use them if circumstances permitted) and difficulty is often caused by the uncontrolled nature of the data. Statistics as a subject was developed to meet the need of agricultural and biological research and it needs now to turn to the problem of processing industries

in particular. Box's development of the evolutionary operation[5] is one step in the right direction, but operational research needs a lot more help from the statistician than it is getting at present.

Two techniques of interest to the mathematician are game theory and dynamic programming. The main drawback of the game theoretic approach to problems is that the most straightforward theoretical problem —the two-person zero sum game—seems to have no analogy in industry. The gaming situations with which the operational research worker is confronted are far too complicated for the elegant theory to tackle. On the other hand dynamic programming is a tool of great potential and a study of Bellman's book[6] is commended. The most stimulating aspect of dynamic programming is that it tries to solve the implicit interlocking and mutual dependency of chains of decisions. An alternative approach is currently being made to this problem by Kimball who is developing the theory of Markoff chains.

The manipulation of the mathematical model is still only of secondary importance. Of first importance is the movement into and out of the manipulative language; this is an area in which the skill of the scientist *qua* scientist is paramount. There is a need to develop in teachable form the elements of the diagnosis of an industrial situation. A tendency exists for operational research workers to rush their diagnosis and throw a study into standard queueing, or inventory, or linear programming terms. In some cases not enough effort is put into studying the real life situation and trying to understand it. Equally the relevance of the solution needs to be studied; what may be significant in statistical terms may be nonsense in practical terms. When the facts of life are known the correlation of the birth rate with the immigration of storks becomes nonsense. Equally one needs to know the facts of business or industrial life before proposing theoretical solutions to a practical problem.

5. Organization of Operational Research Groups

The pre-requisite for the success of an operational research group is that its members must have free access to all parts of the industry. As long as this criterion is satisfied and as long as the operational research group is enabled to report direct to all people concerned in the problem being tackled, then the actual location of the group within the parent organization is of secondary importance. One of the problems with which management is faced when setting up an operational research group is that of

establishing its relationship with other specialities. The operational research worker is always quite sure where he ought to be in an organization, namely, reporting to the board of directors. Unfortunately, this claim is also made by the other management specialists. Consequently the reaction of top management is to place its operational research group somewhere in one of its functional departments. The particular department in which the group is located is largely dictated by personalities, but a fairly common location is in one of the functional departments devoted to production or production control. While this does not affect the quality of the work in this particular area it does mean that the group can be inhibited from carrying out studies in other aspects of their firm's concern and it may completely block them from studying top-level strategic problems which occur as a result of inter-departmental conflict. However, location in a production department is probably far better than location in a finance department; some operational research workers find it difficult to get a common language with which to discuss their problems with accountants. Until, as mentioned above, operational research workers and accountants learn a bit more of each other's specialities this language barrier, and possibly also a philosophical barrier, will mean that an operational research group located in a finance department is not likely to flourish.

The most successful location for operational research workers has proved to be either in departments which have no direct functional responsibility or else as an independent entity reporting direct to the top management. This top management can have a direct functional responsibility in the organization, but so long as he is part of a community of top managers who are encouraged to take a broad view of their industry's activity, the operational research worker will be able to survey the whole field of his firm.

Certain exceptions to this generalization can be made in the light of experience to date. The first is that to locate an operational research group within the scientific research department of a company may be to place it under a grave handicap from the beginning. Although the operational research worker is primarily a scientist, the field of problems which he studies is different from those tackled by research men, the time scale will also be different, and the ease of access to the industry itself may be difficult from the pure research side. An operational research group so located may never succeed.

The most important factor in deciding the location of an operational research group in a firm is to enable it to have close relations with other

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management specialities without destroying its distinctive quality. Some organizations have put operational research in the O and M department or inside the work-study department. Such a location is likely to prove inimical to its success in exactly the same way that O and M and work study might also be doomed to failure if they were located within an operational research group. The proper place for the specialists is to exist as separate entities, but this does not mean that there should be no intercourse between them. Indeed in many organizations communication between these specialist functions has been more frank and more useful when they have been kept apart in this way than when one speciality has been organized within another.

One attractive solution to the problem is to have various management specialities on equal terms within a management science department; this would provide the cross fertilization which is necessary and would also enable changes of staff between various specialist techniques to be undertaken so that they may learn something of each other's contributions and methodology. It certainly would make the problem of top management more easy, in that they would not have to resolve the present confusion which often exists in their minds between the proper fields of activity for these various specialities. It is important, however, that the grouping of separate management specialities into a management science department should not cause these separate specialities to lose the stability they at present enjoy by being located within a functional department. Method study for example is often located in the production side and O and M in the finance side; this gives them a sense of belonging. The danger of having a separate management science department is that it might become divorced from the main stream of the industry. One would want to avoid the feeling that those who can manage, do so; and those who cannot manage, do management science.

The Operational Research Society has a responsibility in influencing the relationship of operational research workers with other management specialists. Some method-study engineers and some O and M specialists do not seem clear as to what is operational research, in so far as either its field activity or its techniques are concerned. There should be more joint meetings arranged with other professional bodies. Specialist management services have much to learn from each other; not so that they could do each other's jobs, but rather that the one can where necessary suggest problem areas for the other to tackle. In any event, management will only be confused and antagonized if the facile denigration of each

other's speciality which has become so fashionable in recent years is continued.

6. Education and Training

Many universities and polytechnics have offered courses of one sort or another in operational research. There are four possible audiences for such courses:

(1) management specialists who want to know what operational research is and how it is done;
(2) managers and executives who might want to start up operational research;
(3) scientists who want to know where operational research fits into the general context of the scientific activity;
(4) operational research workers, actual or potential.

If operational research is a science, then it is teachable and the main concern of this section is to establish the type of course needed for this fourth group. It might be that the circumstances to enable a successful teaching of the subject are difficult to obtain, but this does not mean that the subject itself is difficult to teach.

The first point to establish is that it is not sufficient, indeed it is dangerous, simply to teach a list of mathematical techniques. A knowledge of these techniques is necessary, but it is by no means sufficient. One must clearly start with the potential operational research workers at post-graduate level. At present only Birmingham University offers a post-graduate course in operational research and reference to this has been made in a paper by M. G. Kendall[7] which reviews education for operational research. The important basis of post-graduate work is that it must be firmly rooted in practical studies. To do otherwise is like trying to teach chemistry while denying the student access to a laboratory. So far as post-graduate degrees in operational research are concerned one of the best courses is probably that organized at the Case Institute of Technology in the U.S.A. The course for the master's degree takes 2 years' full time study and includes examinations in mathematics and statistics, practical project work for industry, a thesis on a special study and courses of lectures on the scientific method and the specialist techniques of operational research. The Ph.D. course builds on from the M.Sc. course with further work at an advanced level. The great advantage of the post-graduate work is that it is all founded on practical studies

carried out within industry and that the lecturing staff have all had extensive practical operational research experience.

How can this be adapted for use in this country? The prime responsibility rests firmly with industry and courses such as those outlined need money to start them and, more particularly, need experienced staff to act as tutors. Industrial operational research groups should be willing to lend senior staff to universities to help teach operational research and to assist in the research programmes. The selection of these people is a matter of great importance as teaching is a highly skilled profession and the good operational research worker is not necessarily a good lecturer. It will also be necessary for universities to allow their operational research staff and students to do consulting work for industry.

So far as the syllabuses for courses for the first three types of audience listed above are concerned, there is a great variety of content. Some of the courses offered have been quite disturbing. For example, one education institute offered a 2-week "Appreciation Course in Operational Research" which was simply work study, probability theory and linear programming. Fortunately, in the past year the Education Committee of the Operational Research Society has made known its eagerness to help in running operational research courses and one hopes that ultimately standards will be raised. In the meantime, however, some damage has undoubtedly been done to the subject by courses which have been put on by people who have no contact with operational research.

Education in operational research will really have arrived when we have a few chairs founded at our universities; this would ensure the carrying out of the fundamental research which is necessary and would place the teaching of the subject on a firm basis. This will probably only come as a result of the industrial investment in operational research being extended beyond the money necessary to run the industrial operational research groups themselves and on to the founding of university departments.

7. Management Consultancy

One of the major differences between operational research in the United States and in Britain is that whereas in this country the industrial operational research effort is concentrated in a number of large firms, in the United States it extends also on the one side to universities and institutes of technology, and on the other side to management consultants. These consultants include a number of firms of accountants who are taking up

operational research in a responsible and enterprising manner. We in this country are now seeing the initial development of operational research within some of our larger management consultancy firms. Although there are one or two cases where smaller consultant firms are offering operational research to industry merely as a new label to put on their old wares, the major management consultants are in general going into this slowly and cautiously and are appointing experienced operational research men to open up this new field. This stage is one of the most encouraging in recent years. There are also, of course, two or three experienced operational research men successfully working as independent consultants. It is necessary for the development of our subject that these consultant activities should meet with success. We have got to be able to demonstrate to management that investing money in an operational research project is one of the most profitable activities at its disposal.

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8. Careers for Operational Research Workers

During the last few years there has been a boom in operational research. It has hardly been possible to look at a newspaper without seeing operational research jobs advertised. In recent months, however, there have been signs that the boom is coming to an end and that, apart from consultants, there are few fresh jobs being created for operational research men. It will indeed be useful if for a few years operational research can consolidate and the groups established in industry learn about their industries and move on eventually to do strategic studies. There will ultimately be a further expansion of operational research but at the present time many operational research workers are facing the question "what is my ultimate career in this subject?" Operational research depends on an intake of lively intelligent young people. These people are naturally ambitious, in fact if they were not so they would not be of the type for operational research.

Some lip service is paid to the idea that operational research is a training for management. The operational research worker is not *per se* a potential manager. The experience which he collects is obviously the raw material for management experience, but this is not the same as saying he is a manager. However, if an operational research worker is likely management material he will have received first-class training and first-class experience by working in the operational research group. So far very few operational research workers have made the transfer into senior management positions.

Many operational research men, working as they do for top management, wish that they were the top management. It remains to be seen how far this ambition is likely to be achieved. Operational research provides a good career to the £3,000 a year level, but the man who wants to go beyond this is likely to have to do a great deal of thinking to see how far he can translate his operational research experience into something of use in his firm in some other field. Unfortunately, operational research workers are in such demand that some managements are loath to move an experienced man out of his present job into something which will enable him to develop and get more experience, because of the difficulty of replacing the man himself in his present job. During the next few years the careers of senior operational research workers will be watched very critically by the juniors with these points in mind.

9. Conclusions

It is as well at this stage to repeat that this paper is a personal view of operational research and does not necessarily represent the opinion of anyone other than the author. With this in mind the following general points are presented as a summing up of the present position of industrial operational research in Great Britain.

(1) During the war startling successes were achieved by military operational research. This was probably mainly due to the use and acceptance of the subject at the highest strategic and tactical levels. There is, *prima facie*, no reason to doubt that similar successes could be achieved by the strategic use of operational research within industry today. In fact, such work and such successes have already been accomplished. However, most of our work is still in fields which have been the subject of continuing study at the tactical/operational level for many years prior to our arrival on the scene. In such cases management obtains a high level of skill in the operations and it is unlikely that cost reductions or productivity increases of more than 5 per cent to 10 per cent will be obtained by the operational research men. Such proportions may well be highly desirable in absolute terms.

The operational research worker, however, must be patient in understanding the industry in which he works, must not be disappointed that a number of the projects posed him are intractable and, most important of all, he must resist the impulse to over-sell his speciality.

(2) The best way of communicating operational research studies both to operational research scientists and to management is by means of the

operational research journals. Operational research workers need to publish far more than they do at present and so ensure that their work is subjected to lively informed criticism from other scientists. At present the literature not only gives a misleading account of industrial operational research but this research itself is out of balance.

The fields of marketing, purchasing and human problems still await exploration by operational research workers, and until this has been done operational research will be largely inhibited from moving into the field of strategic studies. In this latter field lies the greatest potential of operational research.

(3) Operational research groups and their parent industries have a responsibility to their intake of young scientists; this responsibility extends beyond that of providing an adequate career within an operational research group to that of ensuring that wider experience in the parent concern is provided for those who are likely to benefit from such experience.

(4) The mathematical and statistical techniques need to be developed in the areas of linear programming, queueing theory and inventory control as well as in the broader field of the problems arising from uncontrolled data. There is a need for developing techniques of diagnosis and for assessing the relevancy of theoretical solutions.

(5) There is a need to review the relationship of operational research with other management aids, such as method/work study, organization and methods, accountancy and market research. This needs to be carried out both on the professional level by the appropriate societies and institutions and also by industrial firms which have these aids in their organization hierarchy. One attractive solution to this latter problem is to group some of the aids in a management science department, although certain safeguards would have to be met.

(6) A science will only progress if it has competent scientists working in its speciality. This applies with particular force to operational research and there needs to be an expansion and improvement of educational facilities in the subject. This is particularly needed in the field of post-graduate courses.

(7) A final cautionary note. There is a danger of operational research workers getting too self-conscious and being fascinated by constant reviews of the progress of their subject. While this paper is appropriate to the needs of the Harrogate Conference this year, the author suggests that for the next few years an embargo be placed on further reviews of the subject.

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Some Aspects of a Control and Communication System*

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Aircraft turnround at civil airports is a complex of activities which require co-ordination and control. This paper describes the behaviour, under stress, of a control unit with this function. The symptoms of distress, the diagnosis and the measures prescribed to prevent complete breakdown and to create reserve capacity are discussed. This is a case history.

Introduction

AIRCRAFT are expensive and do not earn money unless they are carrying payload. When the demand for transport is greatest, the time aircraft spend on the ground must be least. When an aircraft lands at the end of one flight, it must be turned round and sent on another in the shortest possible time. Moreover it is important that turnround should be not only quick but punctual, for one late departure can cause others and this can quickly affect aircraft utilization.

Turnround consists of many tasks which must be performed by different units and which must be completed in a set order. If one task begins and ends late, the next is also delayed. Consequently the turnround operation must be carefully monitored and potential causes of delay detected and eliminated quickly. At large airports, therefore, it is important that there should be a control unit which knows at any moment in time the state of preparedness for departure of every aircraft on the apron (or "tarmac") and the likelihood of delay. It would otherwise be difficult to co-ordinate the various activities and, where necessary, make decisions regarding switches of aircraft, or the institution of delay procedures. The co-ordination, monitoring and progressing of turnround activities is undertaken at many U.K. airports, for BEA and associated companies by a BEA Apron Control Section.

In early spring 1956 BEA Apron Control at London Airport began to show signs of overstrain during busy periods. The symptoms were: excessive delay in dealing with incoming information and requests for information, the transmission of erroneous or garbled information, and

* Paper presented at the Annual Conference of the Operational Research Society at Harrogate on 12 June, 1959.

occasional fits of group amnesia when the whole unit forgot to follow up an essential task. Consequently Manager London Stations asked for a study of Apron Control and for suggestions for restoring its efficiency.

The Preliminary Study

The first stage in the study that followed was to find out what Apron Control was intended to do, and what was actually being done.

The purpose of Apron Control was primarily to co-ordinate turnround activities to ensure that flights departed on schedule. It was supposed to do this by continuously checking and assisting the progress of the work of preparing aircraft for flight, and by recording and partly controlling the flow of passengers through the airport building from H.M. Customs onwards. In these respects it was intended to be a progress section. The section's responsibility for the punctuality of departures required that it should also have a secondary function: to obtain the reasons for all late departures, and report these to management.

The co-ordination, or progressing, of turnround activities was effected in the following way. Every aircraft on the apron was visited periodically by an Apron Observer who noted which activities had been completed, and estimated, on the basis of his own experience, which others might cause delay. These facts and opinions were transmitted via VHF radio to Apron Control.

This information was received in Apron Control by the *Departures Controller*, who recorded the completion of activities, and relayed queries regarding potential delays to the sections concerned, received their replies and transmitted these to the Apron Observer. Where a delay in one activity appeared inevitable, the Departures Controller was supposed to decide what rearrangement of turnround activities was necessary and inform all sections of this and the possible extent of delay to departure.

Other sections would raise queries with, transmit information to, and ask for decisions from, the Departures Controller via a direct-line intercommunication system, of which he was the centre, as shown in Figure 1.

That part of the work of Apron Control which related to the flow of passengers through the airport building was carried out by a group known as Airside Gallery Control. This was essentially a liaison group between the Departures Controller and the Passenger Handling Unit. Its members had no authority to control the embarkation of passengers, but had to await instructions from the Departures Controller who, alone,

was in a position to weigh up the relative merits of different tactics. They performed the mechanics of control: that is they made embarkation announcements to passengers, informed the Passenger Handling Unit of impending departures and delays, and so on.

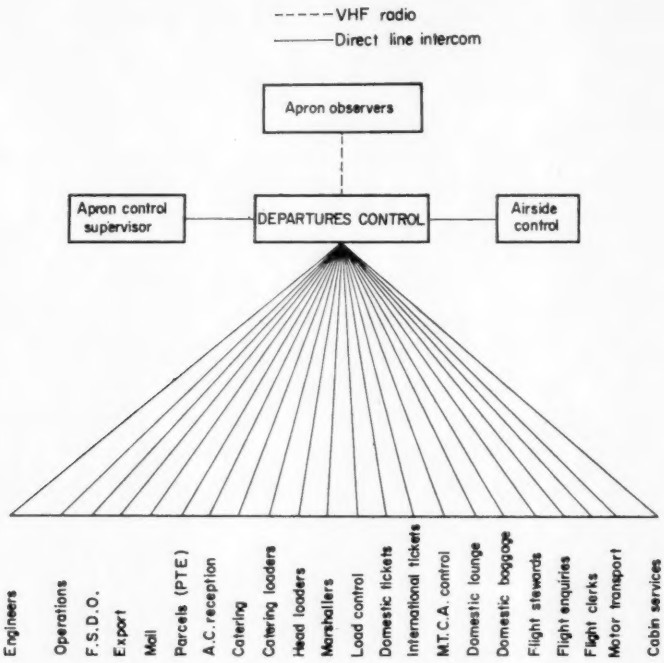

FIGURE 1. *Communication network centred on departures controller.*

The next stage in the study was to identify the symptoms of distress which had occasioned it. These were clearly apparent during busy periods. The first was the difficulty which other sections had of making contact with the Departures Controller to obtain or impart information; for this there was the evidence of long waiting times. The second symptom was the fact that both Apron Observers and the so-called Airside Gallery "Controllers" had to refer all data and problems for decision to the Departures Controller yet experienced great difficulty in getting decisions from him. Third, the Departures Controller garbled and forgot some information, as indicated by the transmission of faulty instructions.

These symptoms, when appraised alongside the known organization of the Apron Control Function and the pattern of the communications

network, suggested very strongly that the Departures Control function was the trouble centre. Consequently the main field study became a study of the Departures Control function.

Study of the Departures Control Function

The Departures Control function, although performed by only one person at a time, was not simple. It is illustrated diagrammatically in Figure 2 which shows information being received over the communication

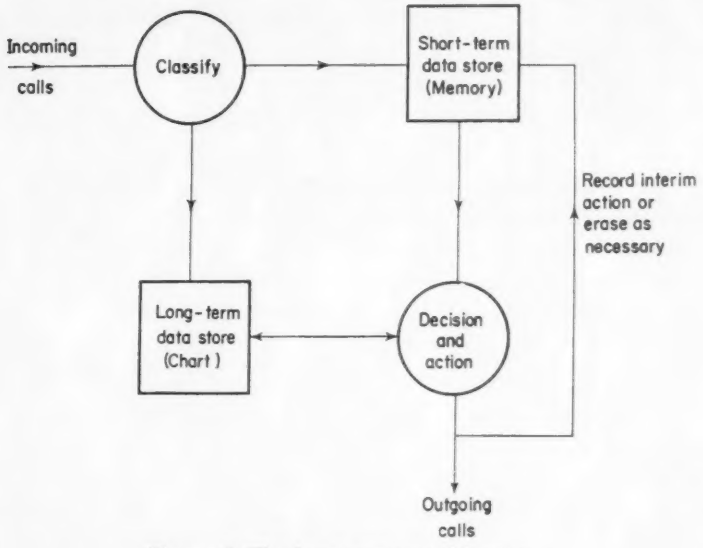

FIGURE 2. *The departures control function.*

system and classified according to its degree of urgency and long-term significance. This is then stored either by being written on to a flight schedule chart on the Departures Controller's desk, or held in his memory, until it is required for making decisions or answering queries or initiating action.

The Departures Controller's memory constituted in effect a rapid access, short-term information store as opposed to the slower access, long-term store and display provided by the chart. A short-term store was required for two reasons; first, as a buffer to hold calls incoming faster than they could be dealt with and, secondly, as an accumulator to collect all calls incoming relating to a specific problem before initiating a decision.

The purpose of the long-term store was to show which important stages in turnround had been completed and to act as a permanent record of delays, and important statistical data (e.g. number of passengers carried).

The first point to be examined was the rate of input of information to the Departures Controller and the time taken to handle it. It was found that during the morning busy period he was occupied in handling calls for more than 65 per cent of his time. Furthermore, call duration was distributed approximately exponentially, with a mean of 0·23 min and, due to a tacit but complex set of priorities, calls were in effect dealt with at random. Queueing formulae showed that incoming calls would be delayed on average by rather more than half a minute and at least 6 per cent of calls would be delayed by more than 2·5 min. Although it was difficult to measure actual delays with any accuracy, small sample checks tended to support these calculations.

It was quite apparent from this point of view alone that the Departures Controller was seriously overloaded, and that either or both of two courses would have to be followed:

(1) Reduce the number of calls to be handled.

(2) Split the job between two or more persons.

Before the number of calls could be reduced it was necessary to see what they consisted of. An analysis of all calls handled during seven peak hours was made, with the following rather odd results.

TABLE 1

SOURCES AND DESTINATIONS OF CALLS HANDLED BY THE DEPARTURES CONTROLLER (1330 calls)

Section or Group	% of all calls
Apron observers	39
Airside control	16
Flight enquiries	8
Operations room	8
Load control	7
19 others	22

Thus 55 per cent of all communication was with other parts of Apron Control, a rather high figure for a section alleged to be co-ordinating other people.

Further analysis showed that:

(1) Forty-three per cent of calls between the Departures Controller and the Apron Observers were reports or queries about the completion of tasks which had in fact been completed so early that no delay could

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have resulted; in other words, were caused by doubts or differences of opinion regarding the times by which tasks should have been completed;

(2) Fifty-two per cent of calls between the Departures Controller and Airside Control were instructions or queries relating to the commencement of embarkation of passengers when in fact there were no hold-ups, and no reason why the passengers should not have been embarked as early as possible; that is, there was no agreement about the time at which passengers should be embarked, or about the type of incidents which would delay embarkation.

The reason why there was so much internal communication within Apron Control seemed to be that each Apron Observer based his work on his personal experience. No two observers interpreted their experience in the same way, so that there was no common basis for decision. Consequently, the only person capable of producing consistent decisions was the Departures Controller who had to be in possession of all information, no matter how apparently irrelevant. This suggested that what was required was a programme which would show, for every type of aircraft, the order in which turnround tasks should be carried out, and the times by which they should have started and finished to ensure punctual departure. It would then be possible for every unit to plan its own jobs in the knowledge that other units would be planning theirs on the same basis, and in such a way that all tasks would fit together. There would no longer be any need for Apron Observers to report to the Departures Controller that such-and-such a task had started or finished. Unless he heard to the contrary, the Departures Controller could assume that the programme was being adhered to. Only if some task began late or met a snag would the observer report to the Departures Controller who would, only then, begin to rearrange the programme for that flight, inform other units of the delay and hold back the embarkation of passengers. For example, Airside Control would automatically, unless told to the contrary, release passengers for embarkation 15 min before flight departure; this would reduce the calls between Airside Control and the Departures Controller by about 50 per cent.

Fortunately the basis of the necessary schedule of turnround tasks was available as the result of some previous work by members of BEA Traffic and Flight Operations Departments and it was possible to put the scheme into effect without delay.

It had been estimated that this scheme would reduce the number of calls handled by the Departures Controller by 20–25 per cent, but in practice

there was a reduction of nearly 30. This served to reduce the incidence of the first two symptoms. Delays in making contact with the Departures Controller were appreciably reduced, and the Departures Controller who had virtually controlled nothing, because he had not time to think about what he should do, was able to act as a restoring force once more.

Meanwhile, the third symptom—garbled or forgotten information—was also studied. The problem here was the capacity of the short-term store (i.e. memory) to retain information accurately. An analysis of the incidence of such errors was made which showed that during busy periods the relative frequency of such incidents was higher. In other words, the probability of a piece of information being distorted or lost increased as the load on the controller increased. The precise form of this relationship was not known, and would have been very difficult to determine, but the limited observations which had been taken suggested that it might be represented approximately by:

$$p(n) = 0{\cdot}005n$$

$p(n)$ being the probability of an item being distorted or lost when n is the input of items per minute to short-term store.

As an input of n messages to the short-term store was equivalent, under the old system, to $1{\cdot}43n$ messages in all to the Controller, this relationship could be rewritten as:

$$\text{percentage of erroneous messages} = p = 0{\cdot}015x$$

where x is the percentage occupancy of the Controller in handling calls.

A relationship of this type is analogous to known results in information theory and, more particularly, psychology. Experiments on remembering have shown that errors increase as the level of activity increases. But they have also led to the concept of memory span, and it was this concept that seemed more relevant, because the amount of information to be stored in the Controller's memory at any one time would appear, intuitively, to depend on the input. Thus if there were a maximum error-free memory span, the significance of high inputs would be that this had been exceeded.

It was necessary therefore to relate the frequency of errors to the amount of information held in the short-term store, and to do this the number of pieces of information stored corresponding to various inputs had to be obtained. A number of observations were made which suggested that the time for which any item was stored was distributed approximately negative exponentially with mean 4·71 min. Consequently the

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growth and decline of the volume of information in the memory could be regarded as a birth-and-death process as follows:

(i) Birth-rate: zero.

(ii) Immigration: random, independent input with mean ν per minute.

(iii) Death-rate: random, independent with mean μN when population size is N.

This process is discussed by Bartlett[2] who shows that the equilibrium distribution, in which interest lay in this case, is Poisson, with mean (ν/μ).

It was decided, quite arbitrarily, that a tolerable error level would be 1 per cent; this corresponded to an input of 2 items per minute to the short-term store. Not all items of information were independent as some related to the same topic, and some merely served to amend all previous messages relating to one topic. For example, one series of 14 calls related to a single incident concerning a piece of freight which would not go into the hold of an "Elizabethan" aircraft. It was estimated that on average there were about 1·6 items of input per topic in store. The number of topics stored, to correspond to an error level of 1 per cent, was therefore on average

$$\frac{4{\cdot}7 \times 2}{1{\cdot}6} = 5{\cdot}88.$$

Thus it could be said that the effective capacity of the Controller's memory was about six topics at a time over long periods. Occasionally the instantaneous content of his memory would, of course, be as many as 10 or 11 topics.

This was a very approximate, empirical means of determining how much information the Departures Controller should be expected to hold in his memory, and no magical general validity is claimed for it. But it served as a guide in deciding what to do. There was no doubt however that from the point of view of accuracy, the Controller had been overloaded, during busy periods, as 2 calls per minute on the short-term store was equivalent to 2·86 per minute in all, and this corresponded to an occupancy with calls of about 65 per cent, which had been continuously exceeded during busy periods. Under the new system, however, with fewer calls being handled, the trouble would be, and in fact was, alleviated for a while. Nevertheless, it seemed that, consideration of waiting times apart, even with one controller it would be desirable to find some way of storing some of the short-term data differently, but still conveniently enough to permit rapid writing-in, rapid access and easy erasure. Under the revised scheme whereby only information on deviations from a set programme was to be passed by the Apron Observers, the major part of short-term

information related to the source of impending or current delays and the state of readiness of the aircraft. Merely to write information on to a chart like the long-term store would be unsatisfactory, because writing is slow, notes are too cryptic always to be easily deciphered and erasure is slow.

Consequently a rapidly assembled and erased, visual display was required. But it was common sense that such a display could not be designed until it was known how many people would have to share it. It was necessary to assess the adequacy of one Departures Controller in the future, and the rate of handling of information was considered afresh.

A series of experiments of various sorts showed that it did not seem possible to reduce the call holding time appreciably. The next stage, therefore, was to see whether one Departures Controller would suffice in future years.

Statistical analysis of call rates showed that under the new system the number of calls handled per flight was proportional to the number of flights scheduled to depart during the next hour, and equalled 9·5 calls per flight (the distribution was not binomial). It was known that the number of flights departing during the peak morning hour of each summer day in 1960 would be about 22, so that the average time taken to handle peak hour calls would be:

$$22 \times 9{\cdot}5 \times 0{\cdot}23 = 48{\cdot}1 \text{ min.}$$

This would represent an average occupancy of about 80 per cent, entirely made up of calls requiring action. One man would not be able to cope with this work adequately, and it would have to be split between two people.

This conclusion reinforced the argument that the short-term storage of information should be altered, for two men could share a common, written, long-term information store—although the visual display would have to be carefully designed for both to be able to write into it and read from it—but they could not share each other's innermost memories and make use of a common short-term store, based on memory only. Without some easily manipulated and interpreted visual display, as mentioned previously, and without some specific division of responsibilities between them, they would either have to spend much of their time exchanging information verbally, or record every minor detail of information on the visual display. In either event, the rate of working of each would be less than that of the single Departures Controller. The capacity of two men would be only about one and three-fifths times as great as the capacity of one; this suggested that the work should be divided between them in

such a way that the amount of short-term information used by both should be minimal, and that way this should be of a simple type.

The more obvious divisions of responsibility, such as for one controller to be responsible for domestic, and the other for international flights, did not provide an even distribution of work. Eventually it was found that the most suitable split was by time and function. This scheme required one controller to deal with all functions which must be completed before 20 min prior to departure of each flight, and the other to deal with those begun after that time. There was almost no overlapping of functions at 20 min before flight departure and the first man could complete his task for each flight, record data of long-term significance on a Flight Card (which replaced the old charts) and hand each flight and its card over as a unit package to the second.

This division of responsibility would be quite even, because the more serious, because more urgent, snags occur near flight departure. Although Controller A would be dealing with as many as 40 flights at one time, whereas Controller B would not be concerned with more than about 8, the actual number of calls being handled per minute, and the number of topics being stored by each in his memory, would be almost the same, and both well below the respective critical levels.

The transfer of most of what short-term information of common interest which there remained from memory to a display was studied by a work study colleague, Mr. E. Buesnel, who solved the problem by:

(i) providing a small (6 in. × 3 in.) data card for each flight which could be
(ii) clipped on to a metal plate, and
(iii) all plates would fit into, and slide along a display channel on a
(iv) new, ergonomically designed, two-position controller's desk;
(v) short-term data were recorded by the use of coloured, magnetic discs which could be quickly put on to the flight cards and quickly removed. A colour code was devised to indicate different types of delay—e.g.

Blue disc —catering
Black disc —engineering
Blue and yellow—catering—passengers held in airside lounge.

This procedure was proposed and adopted. (Subsequently the controllers have modified and improved the coding themselves.) A short check carried out later indicated that errors had almost completely vanished, and that there was no difficulty in the sharing of the display.

The New Departures Control System

It may be useful to summarize the results of this study. It ensued that during normal weather conditions there was a unanimously accepted programme which units could follow and deviations from this programme could be detected and fed back to a control. Here action could be taken to restore the programme or minimize the effect of deviations and where the deviations, their causes and their effects could be adequately recorded for later analysis and system improvement.

Previously there had been no clearly defined programme; relevant and irrelevant information was confused and fed to the controller and the short-term store (i.e. memory) had been overloaded. These deficiencies were remedied; the single, overworked control function was replaced by two almost independent functions and the conditions necessary for this to be done successfully were determined and later verified.

Although the words "servo mechanism", "negative feedback" and "control loop" have not appeared in this paper, it can be seen that Apron Control is essentially a means of providing negative feedback to keep a system in equilibrium, and that this study was aimed solely at designing a more effective way of doing this.

The new system based on this study proved very successful when introduced during normal weather conditions, when the order of flight departures (i.e. the schedule) was not disputed; during bad weather conditions, such as fog and ice, when the order in which flights depart is no longer that shown on the schedule, but is the resultant of many forces, not all inhuman, the system was less successful. It was partly as a result of this that a more comprehensive study of the organization of flight departure was called for and this later study is still in progress. This study however has provided the stimulus for Mr. Longton's paper[3] at this Conference.

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Economic Batch-size Determination for Multi-product Scheduling*

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In the determination of batch-sizes in batch production, different criteria can be used as guiding principles, depending on the specified objectives. In a production schedule consisting of several products, an optimum solution for the whole production schedule is sought. At the same time it is necessary to ensure that the total production costs of each individual product will not exceed a certain pre-determined value. Solutions for overall optimization of the schedule with respect to maximum profit per batch, or with respect to maximum return on the total cost of production of the batch, have already been published. This paper suggests a method for determining the economic batch-sizes when it is desirable to maximize the rate of return per unit time for a multi-product schedule.

Introduction

In determining optimal batch-sizes in batch production the question of clearly defining the objective is a fundamental issue. In a previous paper[1] the author pointed out four criteria that can be considered:

(a) *Minimum total production costs per unit*—this is the criterion commonly found in literature. The costs per unit are usually described by the mathematical model

$$Y = \underbrace{c}_{\text{constant costs}} + \underbrace{(s/Q)+KQ}_{\text{variable costs}} \tag{1}$$

The constant costs per unit are those that are virtually independent of the batch-size. The variable costs are a combination of a hyperbolic (set-up charges) and linear (carrying charges, including interest and storage) terms. Minimum costs per unit are obtained when $dY/dQ = 0$, yielding a batch-size

$$Q_m = \sqrt{(s/K)}. \tag{2}$$

(b) *Maximum profit for the batch*—the term "profit" is sometimes used rather loosely. Profit per unit is evidently a maximum when total production costs are at a minimum, provided the sales price is fixed independently of the production costs. Businessmen, however, often talk about maximizing the profit for the whole operation, which is sometimes interpreted to

* Based on a paper presented at the Fourteenth National Meeting of the Operations Research Society of America at St. Louis, Missouri, U.S.A., in October, 1958.

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mean profit for the whole batch. It is not difficult to see that when this interpretation is adopted, the two criteria cease to be synonymous. The profit function for the batch is described by

$$Z = Q(Y' - Y). \tag{3}$$

As the cost function Y is rather flat in the vicinity of the point of minimum, the total profit of the batch is expected to increase when a batch size larger than Q_m is selected, and it can be shown that maximum profit is obtained when

$$Q_p = p' Q_m. \tag{4}$$

It is clear that maximum profit is realized at higher production costs than those corresponding to the minimum point on the Y curve. This, however, is not the main objection to the criterion of maximum profit per batch, since the increase in production costs due to such policy is likely to be comparatively small. But large batches mean large mean stocks, large amounts of capital tied up and low turnover.

(c) *Maximum return*—return is defined as the ratio of profit to the cost of production. In this way the criterion of maximum profit is modified to include some measure of the effort that has to be invested in the operation. Capital is one of the limited resources of the enterprise and it is, therefore, desirable to have the highest output (profit) per unit of input (cost to produce). By this definition

$$\text{Return} = \frac{\text{Profit}}{\text{Cost of production}} = \frac{Q(Y' - Y)}{QY} = \frac{Y'}{Y} - 1.$$

When the sales price is independent of the batch-size, it is evident that maximum return is attained when Y is minimum, i.e. when Q_m is selected. Thus the goals of minimum costs per unit (criterion a) and maximum return (criterion c) are compatible with each other and yield the same result for the optimal batch-size.

(d) *Maximum rate of return*—there is another important factor that has to be considered, namely time. Of two projects yielding the same return, the one having a shorter consumption period would be preferable. It would allow the enterprise to realize this return more quickly and use the capital on some other projects. The rate of return is defined as return per unit time and if a batch is consumed in a period of time T (this being the cycle time for that particular product), the rate of return for the batch is

$$R = \frac{\text{Return}}{\text{Cycle time}} = \left(\frac{Y'}{Y} - 1\right)\frac{1}{T} \tag{5}$$

where the cycle time $T = Q/a_e$. If one starts with the minimum cost batch-size Q_m, and provided the sales price Y' is fixed, one may expect the rate of return to increase when a slightly smaller batch-size is selected, since the cycle is thereby reduced, whereas the production costs Y do not increase appreciably in that region. Eventually a maximum rate of return is attained when $dR/dQ = 0$ and it has been shown[2] that at this point

$$q^4 - 4p'q^3 + 2(1 - up')q^2 + 2uq + 1 = 0. \qquad (6)$$

The batch-size corresponding to this criterion is $Q_e = qQ_m$.

A full discussion of these criteria is given elsewhere,[1] but it may be useful to compare briefly maximum rate of return and minimum costs per unit. It should be noted that the conventional criterion of minimum costs per unit does not take count of three important parameters: the sales price; the relation between input (allocation of facilities, cost of producing the batch) and output (profit); the time factor (which determines the rate of turnover). All these parameters are included in the solution for maximum rate of return, in spite of the fact that this batch would involve higher production costs than those at the point of minimum (furthermore, since $Q_e < Q_m$ the batch-size Q_e is on the steeper portion of the Y curve, where the costs are more sensitive to changes in the batch-size than in the region $Q > Q_m$). Since maximum rate of return ensures lower average stocks and higher turnover by taking account of relevant parameters, it is a better criterion to adopt, unless:

(i) Competition is severe and cuts in sales price may be expected—in this case the enterprise should endeavour to produce at minimum costs in order to avoid a loss.

(ii) Better production methods and reduced costs may be expected with the production of larger batches.

Under such circumstances the minimum cost batch-size may be preferable. It is evident that the question of a criterion selection is an extremely important issue and has to be carefully and thoroughly analysed, since the adoption of a specific criterion may have serious implications of stock levels, production schedules and effectiveness of operations as a whole.*

* That optimal batch-sizes will greatly differ by selection of different criteria can be demonstrated by the following example: $p' = 1{\cdot}6$, $u = 20$ and $s/K = 100{,}000$, the optimal solutions derived by the different criteria would be:

Min. costs per unit batch Max. return batch	$Q_m = \sqrt{(s/K)}$	= 10,000 units
Max. profit batch	$Q_p = p'Q_m$	= 16,000 units
Max. rate of return batch	$Q_e = qQ_m$	= 6,700 units

Optimizing a Production Schedule

In multi-product manufacture, the production schedule has to be constructed in a manner that will ensure availability of the products in the stores, i.e. the production of a certain product has to start before its stock is completely depleted. At the same time, building up excessive stocks is clearly undesirable. The model considered here consists of n products produced in sequence, in other words the same equipment, man-power and other facilities are used to produce each product in turn. The production schedule would read as follows:

Set-up time for product 1
Production period to make Q_1 units of product 1
Set-up time for product 2
Production period to make Q_2 units of product 2
Set-up time for product 3
etc.

The length of the production cycle = total set-up time for n products + total production time for n products, and at the end of this cycle it is product 1's turn again. If the consumption period of the ith product is T_i and as $T_i = Q_i/a_{ci}$ it has been shown that

$$\left.\begin{aligned} T_1 = T_2 = \ldots = T_i = \ldots = Tn \\ \text{or} \qquad Q_i = \alpha_i Q_1 \end{aligned}\right\} \tag{7}$$

under the following conditions:

(i) Consumption rates are deterministic (no uncertainty).

(ii) Stock run-outs are not allowed, i.e. the product must always be available for supply.

(iii) Stock building by accumulation of residuals at end of cycles is not allowed. If, due to scheduling problems, small accumulations of stock do occur, their effect on the model and on the optimal solution is likely to be rather small.

If each product is considered separately, its optimal batch can be computed in accordance with its specific data, but this quantity may last longer than the corresponding batch of another product in the production sequence, and this does not comply with the conditions set above. The purpose of optimizing a schedule is to find an optimum solution for the whole production cycle (by ignoring for a while the individual approach) within the framework of the conditions listed above. This framework implies a quantitative relationship between the batch-sizes of the products included

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in the schedule and makes it possible to express the multi-product problem in terms of one variable, namely the batch-size of product 1.

For the computation of the optimal solution either of the criteria b, c or d listed above can be employed. Optimal solutions with respect to criteria b and c have already been published and the purpose of this paper is to propose a solution in accordance with d, namely a solution for batch-size determination in a multi-product schedule for maximum rate of return. The table in the appendix summarizes the solutions presented so far.

Optimizing a Multi-product Schedule for Maximum Rate of Return

Consider n products, for each of which the rate of production, the rate of consumption (a_c), the costs function (Y) and the sales price (Y') can be established. The rate of return for one production cycle is given by

$$R = \frac{\Sigma Q(Y' - Y)}{\Sigma Q Y} \frac{1}{T}. \tag{5a}$$

Substituting $Q = \alpha Q_1$ (by (7))

and $T_1 = Q_1/a_{c_1}$

the following expression for R is obtained:

$$R = \left(\frac{\Sigma \alpha Y'}{\Sigma \alpha Y} - 1\right) \frac{a_{c_1}}{Q_1} = \left(\frac{\Sigma \alpha Y'}{\Sigma \alpha Q_1 Y} - \frac{1}{Q_1}\right) a_{c_1}. \tag{5b}$$

The rate of return is thus expressed in terms of the batch-size for product 1 and its maximum is found at

$$\frac{dR}{dQ_1} = 0.$$

$$\therefore \frac{\Sigma \alpha Y'}{Q_1^2 \Sigma^2 \alpha Y} \frac{d}{dQ_1}(\Sigma \alpha Q_1 Y) = \frac{1}{Q_1^2}$$

or

$$\Sigma \alpha Y' \frac{d}{dQ_1}(\Sigma \alpha Q_1 Y) = \Sigma^2 \alpha Y. \tag{8}$$

Y is a function of Q, as given by the model (1), so that

$$\alpha Q_1 Y = \alpha Q_1 (c + (s/\alpha Q_1) + K \alpha Q_1) = \alpha c Q_1 + s + K \alpha^2 Q_1^2$$

$$\therefore \frac{d}{dQ_1}(\Sigma \alpha Q_1 Y) = \Sigma \alpha c + \Sigma 2 K \alpha^2 Q_1 = \Sigma \alpha c + 2 Q_1 \Sigma K \alpha^2.$$

Also $\Sigma^2\alpha Y = [\Sigma(\alpha c + (s/Q_1) + K\alpha^2 Q_1]^2$

$$= \Sigma^2\alpha c + \Sigma^2(s/Q_1) + \Sigma^2 K\alpha^2 Q_1 + 2\Sigma\alpha c\Sigma(s/Q_1) + \\ + 2\Sigma(s/Q_1)\Sigma K\alpha^2 Q_1 + 2\Sigma\alpha c\Sigma K\alpha^2 Q_1$$

$$= \Sigma^2\alpha c + \frac{\Sigma^2 s}{Q_1^2} + Q_1^2\Sigma^2 K\alpha^2 + \frac{2\Sigma\alpha c\Sigma s}{Q_1} + 2\Sigma s\Sigma K\alpha^2 + 2Q_1\Sigma\alpha c\Sigma K\alpha^2.$$

Substitute these into (8) and multiply both sides by Q_1^2

$$Q_1^4\Sigma^2 K\alpha^2 - 2Q_1^3(\Sigma\alpha Y' - \Sigma\alpha c)\Sigma K\alpha^2 + Q_1^2(\Sigma^2\alpha c + 2\Sigma s\Sigma K\alpha^2 - \Sigma\alpha Y'\Sigma\alpha c) + \\ + 2Q_1\Sigma\alpha c\Sigma s + \Sigma^2 s = 0. \qquad (8a)$$

Now, when the production schedule is optimized with respect to the return (third criterion), the solution for the batch-size of the first product was shown[1] to be

$$Q_{1M} = \sqrt{\frac{\Sigma s}{\Sigma K\alpha^2}}. \qquad (9)$$

Dividing (8a) by $\Sigma^2 s$, using (9) and the definition

$$q_1 = \frac{Q_1}{Q_{1M}} \qquad (10)$$

the following expression is obtained:

$$q_1^4 - 2\frac{\Sigma\alpha Y' - \Sigma\alpha c}{\Sigma s/Q_{1M}} q_1^3 + \left(2 - \frac{\Sigma\alpha Y' - \Sigma\alpha c}{\Sigma s/Q_{1M}}\frac{\Sigma\alpha c}{\Sigma s/Q_{1M}}\right) q_1^2 + 2\frac{\Sigma\alpha c}{\Sigma s/Q_{1M}} q_1 + 1 = 0.$$

Two non-dimensional ratios are now defined as

$$P_1' = \frac{\Sigma\alpha Y' - \Sigma\alpha c}{2Q_{1M}\Sigma K\alpha^2} = \frac{\Sigma\alpha Y' - \Sigma\alpha c}{2\Sigma s/Q_{1M}} \qquad (11)$$

$$U_1 = \frac{\Sigma\alpha c}{\Sigma s/Q_{1M}}. \qquad (12)$$

The significance of P_1' lies in the fact that when the production schedule is optimized with respect to profit (criterion b), it can be shown that the batch-size for product 1 is

$$Q_{1P} = P'_1 Q_{1M} \qquad (13)$$

which is analogous to (4).

Using the definitions (11), (12), the equation for q_1 is reduced to

$$q_1^4 - 4P_1' q_1^3 + 2(1 - U_1 P_1')q_1^2 + 2U_1 q_1 + 1 = 0. \qquad (14)$$

This equation is similar to (6), which was obtained for one product, only the ratios P'_1 and U_1 differ from p' and u by definition. This means that in order to find the economic batch-size for product 1, i.e. the value of

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q_1, the same methods used for the solution of (6) may be employed, namely:

(a) As equation (14) is non-dimensional, its solution can be presented by a series of curves, as shown in Figure 1. The curves are very useful in that they provide ready answers without having to resort to lengthy numerical solution methods and they can be used both for multi-product and single-product problems. For the solution of a multi-product schedule use P'_1 as the ordinate, U_1 as a parameter and the relation $Q_{1E} = q_1 Q_{1M}$.

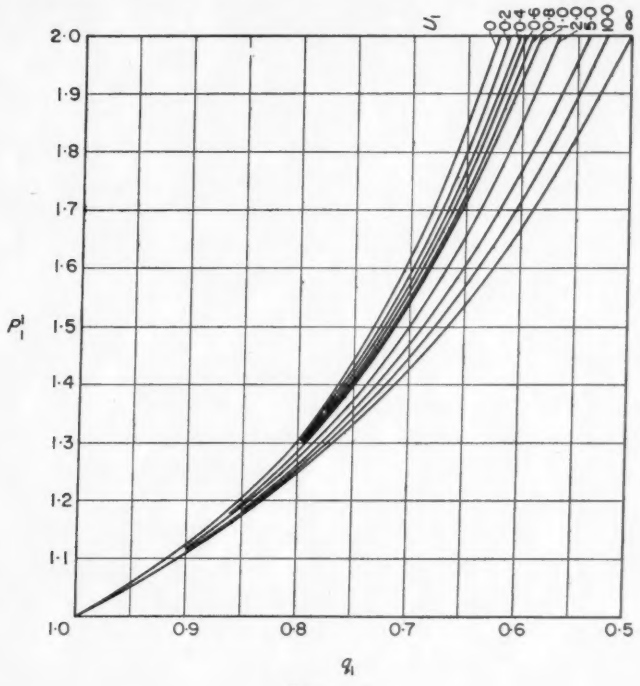

FIGURE 1.

For the solution of a single-product problem, substitute in Figure 1:

p' for P'_1, u for U_1 and q for q_1, using $Q_e = qQ_m$.

(b) The analysis of equation (6) has shown that the solution lies between

$$q = \frac{1}{p'} \quad \text{and} \quad q = \frac{p'^2}{p'} \qquad (15)$$

Likewise, the solution of equation (14) lies between the values

$$\left.\begin{aligned} q_1 &= \frac{1}{P_1'} \\ \text{and} \qquad q_1 &= \frac{p^2}{P_1'} \end{aligned}\right\} \tag{16}$$

which may be used as good approximations for q_1.*

The procedure for determining the economic batches would be as follows:

(1) Start with a data table, in which the values of c, s, K, α and Y' for each product are specified.

(2) Find by (9) the batch-size for product 1 for a maximum return multi-product schedule (Q_{1M}).

(3) Evaluate P'_1 by (11).

(4) To find Q_{1E} use either the approximations (16) or the solutions provided by Figure 1, in which case U_1 should first be evaluated by (12).

(5) Find the other batch-sizes by (7).

Evaluating the Solution

A pre-requisite of a well planned production schedule is that it results in fluctuations of stocks between two pre-determined limits: a high level, which it is undesirable to exceed, and a safety level, below which it is risky to operate. This feature is automatically assured by the above method, since the prescribed quantities and the proper dovetailing of production periods in the schedule are provided at the outset by the condition (7). The proposed schedule has, however, to be examined in two other respects:

(a) Although the solution yields maximum rate of return for the schedule as a whole, it is necessary to ensure that each product is not produced at excessive costs, in other words that the computed batch-size lies within a pre-determined production range. The production range for each product is defined as the range of batch-sizes for which total production costs

* It can be shown that the first approximation applies when $U_1 \to \infty$, and the second when $U_1 \to 0$. This means that the two approximations relate to the extreme curves in the family of curves in Figure 1. As in reality U_1 is between 0 and ∞, the optimal solution would lie between the two curves and it is evident from Figure 1 that the difference between results derived from the two approximations is small for relatively high values of q_1. When, for instance, $U_1 = 5{\cdot}0$ and $P_1' = 1{\cdot}9$, the first approximation yields $q_1 = 0{\cdot}77$, the second yields 0·81, while the actual answer is 0·78. The error, however, increases as P_1' increases.

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will not exceed a value dictated by management policy. It is convenient to state this policy by specifying the maximum acceptable value of p, which is a measure of excess variable costs above the minimum. The ratio p determines the limits of the production range and as long as a batch-size between these limits is selected, the maximum allowable production costs are not exceeded, hence the solution has to pass the so called "p-test".

(b) The proposed production schedule involves a certain production cycle time, which is readily computed when the rate of production is known. It is necessary to examine whether the proposed schedule can be matched by the available facilities of the plant, in other words, whether the production cycle time (governed by the available facilities) and the consumption cycle (given by the solution) are the same. This examination is called the "cycle test".

When the solution does not pass those two tests, it has to be modified, and ways for doing so have been discussed at some length elsewhere.[1] To summarize them very briefly:

(a) When the quantity for a certain product is not in the production range, it may be either:

—too large (i.e. above the upper limit of the production range), in which case the quantity could be divided into two or more sub-batches, so that the product is produced more than once in the cycle; or

—too small (i.e. below the lower limit of the production range), in which case the quantity could be doubled and produced once every two cycles.

It has been shown that the production range is wide enough to enable this method to comply with the difficulty in almost every situation, so that when too large quantities are divided into sub-batches or too small ones are doubled, the resultant quantities will fall within the production range.

(b) When the "cycle test" is negative, then either

—the production cycle is shorter than the consumption cycle, i.e. there is extra time left after the nth product is manufactured and before the first product is due again, so that another product may be added to the schedule; or

—the production cycle is longer than the consumption cycle, i.e. the commitments presented by the schedule are too heavy for the available plant capacity, so that either the capacity has to be increased (use of overtime, acquiring more equipment, sub-contracting) or the commitments have to be appropriately pruned (e.g. by deleting a product from the schedule).

Conclusion

In conclusion it should be noted that the analogy between the single and multi-product problem greatly simplifies the computations of batch-sizes, while the method of adjusting the first solution of the multi-product schedule to individual product limitations and to available facilities remains unchanged. The table in the appendix summarizes the optimum solutions in respect of the various criteria, both for a single and for a multi-product problem.

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REFERENCES

[1] SAMUEL EILON, "Scheduling for Batch Production". Paper presented at the Institution of Production Engineers Harrogate Conference, U.K., July 1957.
[2] SAMUEL EILON, "Economic Lot-sizes in Batch Production", *Engineering*, **182**, No. 4729, 522 (1956).

Notation

K = carrying costs factor.
P_1' = a ratio, defined by (11).
Q = batch-size.
R = rate of return, defined by (5), (5a).
U_1 = a ratio, defined by (12).
T = cycle time.
Y = total production costs per piece, described by equation (1).
Y' = sales price.
Z = profit for the batch, given by (3).
a_c = consumption rate (units per unit time).
c = constant costs per piece.
p = the ratio $\dfrac{Y-c}{2KQ_m}$
p' = the ratio $\dfrac{Y'-c}{2KQ_m}$
q = the ratio $\dfrac{Q}{Q_m}$
s = preparation costs, including machine set-up.
u = the ratio $\dfrac{c}{KQ_m}$
z = return, defined in the appendix.
= the ratio $\dfrac{a_{c_4}}{a_{c_1}}$

Suffixes:

1,2,...*i*,...*n*—relating to products 1,2,...*i*,...*n*.

E—relating to the batch in an optimized multi-product schedule, for maximum rate of return.

M—relating to the batch for maximum return in an optimized multi-product schedule.

P—relating to the maximum profit batch in an optimized multi-product schedule.

e—relating to the batch for a single product optimization, for maximum rate of return.

m—relating to the minimum cost batch for a single product optimization.

p—relating to the maximum profit for a single product optimization.

Appendix

TABLE 1

SUMMARY OF SOLUTIONS

Objective	Single Product		Multi-product	
	Function to optimize	Solution	Function to optimize	Solution
Minimum costs per unit	$Y = c + \frac{s}{Q} + KQ$	$Q_m = \sqrt{\frac{s}{K}}$	— (meaningless)	—
Maximum profit per batch	$Z = Q(Y' - Y)$	$Q_p = p' Q_m$	$Z = \sum Q(Y' - Y)$	$Q_{1P} = P' Q_{1M}$
Maximum return	$z = \frac{Y'}{Y} - 1$	$Q_m = \sqrt{\frac{s}{K}}$	$z = \frac{\sum QY'}{\sum QY} - 1$	$Q_{1M} = \sqrt{\frac{\sum s}{\sum K\alpha^3}}$
Maximum rate of return	$R = \left(\frac{Y'}{Y} - 1\right)\frac{1}{T}$	Equation (6) Graphically: . Figure 1 Approximations by (15)	$R = \left(\frac{\sum QY'}{\sum QY} - 1\right)\frac{1}{T}$	Equation (14) Graphically: Figure 1 Approximations by (16)

Business Games in Business

H. R. WATKINS
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A business game which does not need an electronic computer has been described by Andlinger.[1]

As a result of experience gained from introducing a game as part of a training course for middle management since May 1958, some fundamental changes have been made. The purpose of this article is to compare results obtained from the original and modified versions and to describe the latter in detail.

While the essential characteristics of the Andlinger game are described, reference to the original paper is recommended.

1. Introduction

BUSINESS games attempt to simulate a business situation so that executives who are being trained can be placed in this environment and be called upon to cope with the sort of problems which occur, admittedly in a much more complex way, in real life. The time allowed for making decisions is limited for two reasons; first in order to sustain an impression of working under pressure, and secondly since the time available for the game as usually played is necessarily short. This means that after decisions have been taken the implications must be worked out as quickly as possible and this has led to the existence of two basically different types of model. In each, that part of the game manifest to the players is essentially the same—individuals are grouped into three or four teams representing companies all selling the same product in competition with one another, and these companies are required to take such decisions as where to market the product, where and to what extent to advertise, whether to support a research development programme, and so on. We are here considering the general form of business game and excluding specialized models developed for a particular industry or group.

It is in the means by which the implications of the decisions taken by the teams are worked out that the two models differ. In games using an electronic computer, such as those developed by the American Management Association and IBM, relatively complex equations are used to evaluate the results of such decisions. These equations express the "attractiveness" of the product as a result of the price set, money spent on advertising, research and development, etc. and so arrive at a market share. In a similar way production costs are also established.

The second type of model does not require a computer and it is this with which we are concerned here; since a quick evaluation of results is essential, a much simpler method of evaluation is needed. A unit of time (in both types of game) is 3 months, and in practice the implications of the decisions taken by the teams can be worked out in about 5 min; allowing a further 10 min for discussion by the teams means that a year's trading can be covered in an hour.

The game we have used is developed from that described by Andlinger,[1] and reference should be made to this for a full account. In the light of experience gained from a considerable number of sessions held with training courses attended by middle management we have made a number of changes in the structure. This brings out the fact that the model is a flexible one in that it can be readily modified to meet different viewpoints.

2. *The Game*

An outline of the game as it is now played by four teams each of four members will be given; only where the original has been considerably modified will this be indicated, minor variations being apparent after reference to the paper quoted.

(a) *Equipment*

A Control Board (Figure 1) is used by each team, the left-hand side being divided to represent twenty-four sales areas, the right-hand side operations, and coloured squares of cardboard are used to record the movement of salesmen, the progress of items through production, etc.

The other record is a Quarterly Summary (Figure 2) which provides a company each quarter with a statement of its financial position. This is completed by the umpire located with each team, and in order that the umpire knows what decisions his team have taken, the company is required to complete a Decision Form (Figure 3) in duplicate each quarter recording their proposals, the amounts of money involved and the sales areas to be visited by salesmen.

The Decision Form has been introduced for two reasons: (a) to avoid frequent delays in the game caused by incorrect entries made if the Quarterly Summary is completed by the team—an experienced umpire can produce a Summary much more efficiently and (b) to correct the tendency for a member who completes the Summary, and who thus possesses all the information, to dominate the team.

(b) *Time Periods*

Each period represents 3 months. Decisions are recorded on the forms handed at a given time to the umpires and the resulting sales determined as described later in Section 4(b). One copy of the Decision Form is

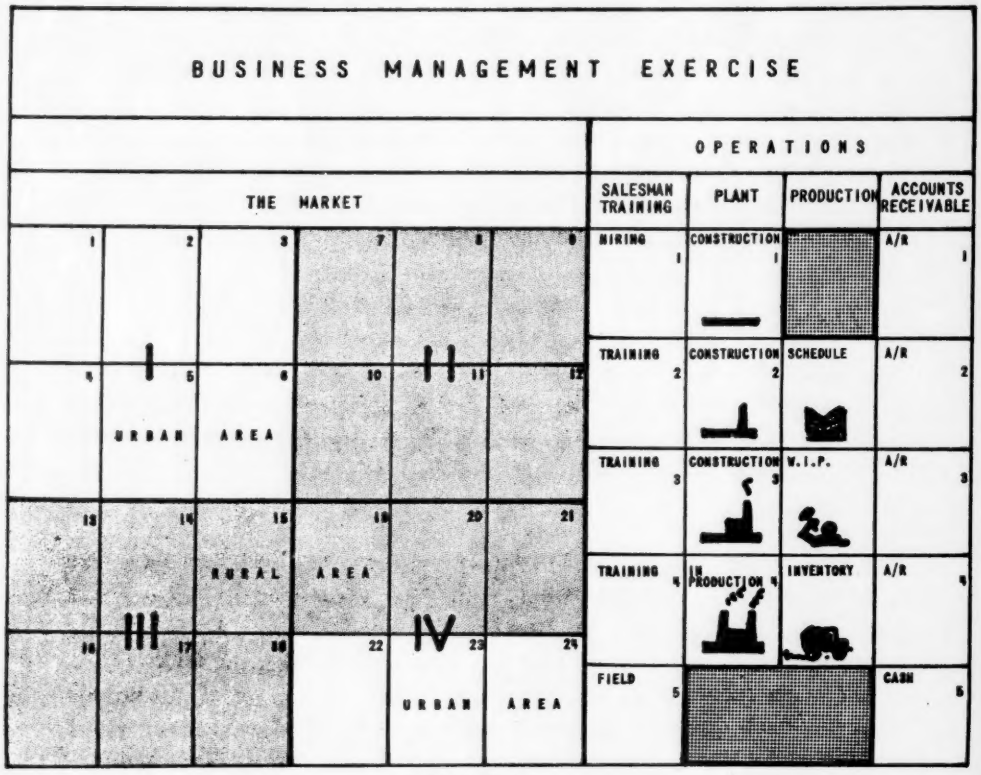

FIGURE 1.

returned immediately, so that while the current Summary is being prepared by the umpire a team can begin to decide its actions for the next quarter. The completed Summary is returned, a new deadline is set and the cycle begins again.

For the first year of operations, when the companies are setting up in business and no sales are possible the umpire sits in with the team on a purely reference basis and helps to complete the records for the first four quarters; it takes about 40 min after a preliminary briefing to reach

the stage where the Decision Forms for the first quarter of the second year have been completed and the game then continues quarter by quarter.

BEAVER COMPANY QUARTER 1 YEAR 5

SALES
Indicate calls by ticks, so — ☑

1 ☐ ____
2 ☐ ____
3 ☐ ____
4 ☐ ____
5 ☐ ____
6 ☐ ____
7 ☐ ____
8 ☐ ____
9 ☐ ____
10 ☐ ____
11 ☐ ____
12 ☐ ____
13 ☐ ____
14 ☐ ____
15 ☐ ____
16 ☐ ____
17 ☐ ____
18 ☐ ____
19 ☐ ____
20 ☐ ____
21 ☐ ____
22 ☐ ____
23 ☐ ____
24 ☐ ____

TOTAL ____

PROFIT AND LOSS STATEMENT

	Units	£	
SALES	70	—	28.000
INVENTORY	—	—	
At beginning	98	14.700	
Current Prods.	100	15.000	
Total	198	29.700	
Less Units sold	70	10.500	10.500
At end	128	19.200	
Gross Profit			

EXPENSES

Hiring Salesmen		
Salaries	2.800	
Advertising	3.600	
Research & Dev.		
Calling in A/cs.		
Staff Work		
Total		6.400
NET PROFIT		11.100

DISPOSITION OF CASH

Beginning Cash	15.000	
A/R Collections		
Normal	26.400	
Called in		
Total Income		41.400
Total Expenses	6.400	
W.I.P. Cost	15.000	
Plant Invest.		21.400
Total Outgoing		20.000
ENDING CASH BALANCE		

WORK IN PROGRESS

Fixed Cost	6.000
Variable Cost ( )	9.000
Total W.I.P.	15.000

TOTAL ASSETS

Cash	20.000
A/cs Receivable	110.400
Units - Inventory	19.200
W.I.P.	15.000
Plant	72.000
TOTAL ASSETS	236.600

FIGURE 2.

(c) *The Market*

The companies are trying to sell units of capital equipment at a price of £400 in a market which is divided into twenty-four sales areas. Each of these has a certain potential, such that in every quarter customers in some areas are prepared to buy a number of units, while others may buy none. A customer who buys in one quarter is normally prepared to buy in the next; if the market is a growing one, the tendency will be for the total

potential to increase, while in a recession sudden falls in potential may occur. When the companies begin operations the market is an expanding one. If a company wishes to know what the total market potential is at

DECISION FORM

COMPANY BEAVER

QUARTER 1 YEAR 5

SALES

Indicate calls by ticks, so ✓

	Reg.
✓ 1 7	I
✓✓ 2 8	
• 3 12	
✓ 4 7	
✓✓ 5 12	
✓ 6 7	
7	II
8	
9	
10	
11	
12	
13	III
✓ 14 11	
15	
✓ 16 6	
17	
18	
19	IV
20	
21	
22	
23	
24	

TOTAL SALES 70

Position at end of last Quarter

Cash 15.000

Accts. Rec. 108.800

Inventory 198 Units

W.I.P. 100 "

Scheduled 80 "

The following decisions have been taken:-

Plant

To construct initial plant

To put in ..... extra line(s)

To schedule 100 units

Expenses		Amount
Engaging ..... salesmen		
Salaries for 7 salesmen in the field		2.800
Advertising		
Region I	3.600	
Region II		
Region III		
Region IV		
	3.600	3.600
Research & Development		
Calling in Accounts (Amount .....)		
Market Information		
.....		
.....		
.....		
Total Expenses		6.400

ONE SALESMAN RESIGNED

FIGURE 3.

any time, that is, the total number of units that all the customers would together buy in a given quarter if all possible sales were made, they may buy the information from the umpire for £800.

The sales areas are shown on the left-hand side of the Control Board and it will be seen that they are divided into four regions each containing six These regional groupings enable market research and advertising to

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be carried out in one region at a time. If a company wishes to know the potential of each of the six sales areas in a region, that is, the number of units that would be bought in each area in a given quarter if all possible sales were made, then they may obtain this information from the umpire by paying £8,000. This information is of use in determining where are the worthwhile customers, and as it is assumed that any work of this sort carried out in a region would stimulate interest in the product, a company having this market research performed in a region has an enhanced chance of making sales in that region for the quarter concerned. Furthermore a company can obtain an up-to-date review of any region they have previously had surveyed for a further £800 in a quarter.

It will also be seen that the twenty-four areas are grouped into one rural and two urban districts; the significance of this is that in an urban district a salesman can cover an area twice in each quarter while in a rural district he can make only one call.

At the end of each year information is available as to the amount of the total market (actual sales) that each company has been able to capture. A company wishing to find out its own position may do so for a fee of £800.

Market research has been made more attractive while at the same time being sufficiently costly to need careful appraisal; it now enables a continual check to be kept on an area if a company so wishes.

(d) *Salesmen*

A salesman costs £4,000 to engage, £400 a quarter in salary and takes a year to train; there is a possibility that a salesman will resign. (This is a 5 per cent probability and a random number table is used each quarter to see whether any of the company's salesmen are affected; this does not apply to salesmen in training.)

(e) *Advertising*

Allocations for advertising are made by regions in quotas of £1,200 and up to five such quotas may be placed in any region in any quarter. It is effective only for the current quarter and in the region or regions nominated.

(f) *Research and Development*

A company which has been able to develop a superior product has a better chance than its competitors of making sales. In order to try for this advantage it is necessary to spend money on a research and development programme; unlike advertising which is effective only in the current

quarter, research and development is regarded as a long term project and the chance of getting an improved product increases with the total amount of money which a firm has invested in research and development work. A minimum amount of £4,000 a quarter may be invested, but there is no upper limit and any sum may be allocated each quarter providing it is a multiple of £4,000. Depending on the total so far accumulated there is a certain probability of getting an improved product; if such an improvement is achieved the company is immediately notified and all units *scheduled* for production in that quarter are assumed to be equipped with the improvement. When these units go into the market they will have a better chance of being sold than those of their competitors.

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(g) *Production*

Each company must build initial plant at a cost of £60,000 and this will produce up to 50 units/quarter. If a greater throughput than this is required, extra production lines may be added at a cost of £12,000 each, and each line will increase the maximum output by another 50 units. Thus with the original plant units (in batches of 10) from 0 to 50 may be produced each quarter, with an extra production line from 0 to 100, with two extra lines 0–150, and so on. Units can only be produced in batches of 10.

As soon as a company decides to start construction it must pay the cost of the plant, and the time taken to complete construction is 3 periods (i.e. 9 months). The decision to build plant is shown on the Control Board by placing a counter on the first square under "Plant"; at the same time the company is debited with the cost of the plant. In the following quarter the counter is moved to the next square down while in this same period the company can schedule the number of units to be produced by this plant when it is in production 6 months later. For the next quarter (the third and last construction period) these scheduled units become "work in progress" and others can be scheduled. Finally, in the next quarter the plant is "in production" and the first lot of units are made and are shown as being "in inventory", available for selling. It will be seen that once the company is in full production there is a constant flow of counters representing units down the board. Units are scheduled for 6 months ahead, pass on to work in progress and then to inventory and are removed from here as they are sold. A company must therefore always be planning at least 6 months ahead to co-ordinate sales and production; if it wishes to increase its maximum capacity it must of course look 9 months ahead.

The total production cost of units, which is incurred at the work in progress stage, consists of a fixed and a variable element. The fixed cost is incurred each quarter regardless of the number of units made and even if none is produced. With the initial plant which has a maximum capacity of 50 units this fixed production cost is £3,000 per quarter. The variable cost per unit is £100. Thus at maximum output of 50 units/quarter the total cost is given by:

Fixed cost	£3,000	
Variable cost	£5,000	(50 × £100)
Total	£8,000	

an average cost per unit of £160.

If only 30 units were made in a quarter the cost would be:

Fixed cost	£3,000	
Variable cost	£3,000	(30 × £100)
Total	£6,000	

an average cost per unit of £200.

If in order to increase capacity extra production lines are started up (each at an initial cost of £12,000) then fixed costs rise and the variable cost per unit decreases. A company thinking of increasing its production capacity can find out from the umpires exactly what these new costs would be in return for a payment of £800; alternatively it can decide to add another line without knowing the new production costs in advance, and it will then be given the information when the new line reaches the work-in-progress stage.

When units are made and are put into stock they are added to the inventory at actual cost, i.e. the cost figure as worked out in the examples above. When a unit is sold, it is removed from the inventory at the average cost of the items in stock (i.e. the total value of the inventory divided by the number of items in it).

(h) *Finance*

The management of the company's finances requires considerable thought, particularly in the early stages. Each company starts with £160,000 out of which minimum plant must be bought for £60,000 and the business can be built up only by skilful reinvestment. A unit is sold for £400, and as a result of the sales in a quarter a sum of money, the

"accounts receivable", will become due for collection a year later. This is due to the fact that the competitive nature of the business has led to customers being given a year's credit. This can be shown on the Control Board by writing the amount on a counter and placing it in the first square under "accounts receivable"; each quarter the counter is moved down one square until after four quarters it moves into the square marked "cash" and the money is then available for reinvestment. It will be appreciated that when a company first starts up in business it is a year before salesmen can take the field and then another year before the money from their first success comes to hand, so that a company must reckon to operate for 2 years without any fresh capital.

If a company finds itself short of cash it may call in outstanding accounts receivable (the oldest accounts first) to get money immediately; this must be done in units of £400. The cost of doing this is charged at 20 per cent of the amount called in.

By asking the umpire at the end of the 5th, 9th, 13th, etc. quarters, a company can obtain free of charge a statement of the total assets at year end for each of their competitors. This is equivalent to the information obtained from annual statement.

(i) *Summary*

When the exercise is in full swing its dynamic nature should be clear. From a production point of view, the Control Board will carry counters indicating units scheduled, work in progress, and the number in inventory; at the end of each quarter the results of salesmen already in the field will lead to some of the units being removed from inventory, other units will go into inventory from work in progress, the scheduled work will move to work in progress and more units must be scheduled for production. Financially there will be a steady stream of accounts falling due each quarter, and the money so obtained will be ploughed back in the shape of further production. There are also the decisions of where and when to advertise, whether to carry out market research, to initiate a programme of research and development, etc. to be taken each time.

3. *The Andlinger Model*

The Andlinger model is a stochastic one and is based on giving each salesman a 25 per cent chance of making a sale; this figure can be increased by advertising or research and development. Thus a page of advertising

will increase the chances of a sale in that region to 29 per cent, two pages to 35 per cent and so on. Similarly an improved product, once obtained, increases all chances by an extra 10 per cent. It also deals with a larger item of equipment selling for £4,000; sales potentials and production quantities are correspondingly smaller.

It is most valuable as a basis for development but is open to the criticism that the effect of chance can play too great a part in determining sales. It is possible for a team to have a run of "bad luck" irrespective of the wisdom of their decisions. It may certainly be argued that this is a fair reflection of life itself and that how a company fares is less important than the opportunities given for discussion and decision-taking, but we feel that in a system designed to operate over a relatively short period such effects should be eliminated. This may be done in a number of ways still using random numbers, but the relationship can also be made a deterministic one in such a way that competition between the companies produces the sales fluctuations.

Before describing how this is done, it may be of interest to compare results arising from typical games played with each model. The graphs display for the Andlinger and modified games respectively (a) total assets (Figures 4 and 5), (b) share of market (Figures 6 and 7) and (c) total sales (Figures 8 and 9). It will be seen from a comparison of the results for (b) and (c) that a more stable result is achieved.

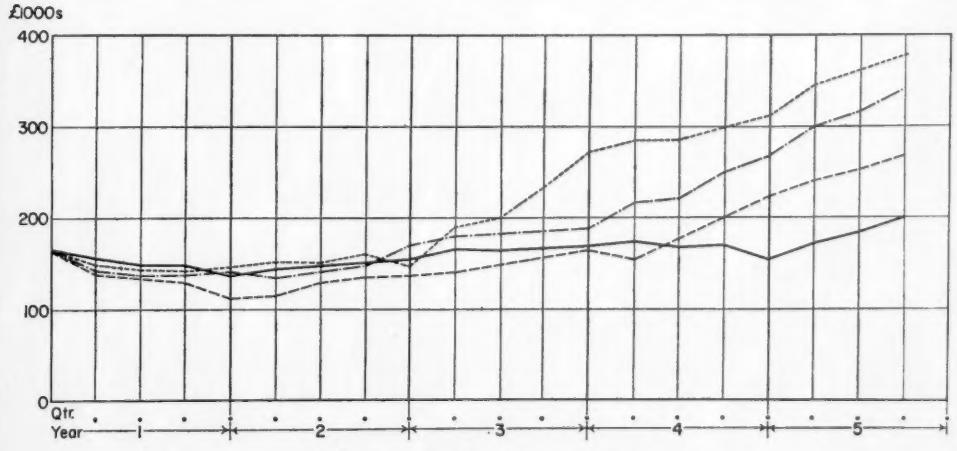

FIGURE 4. *Andlinger game: assets.*

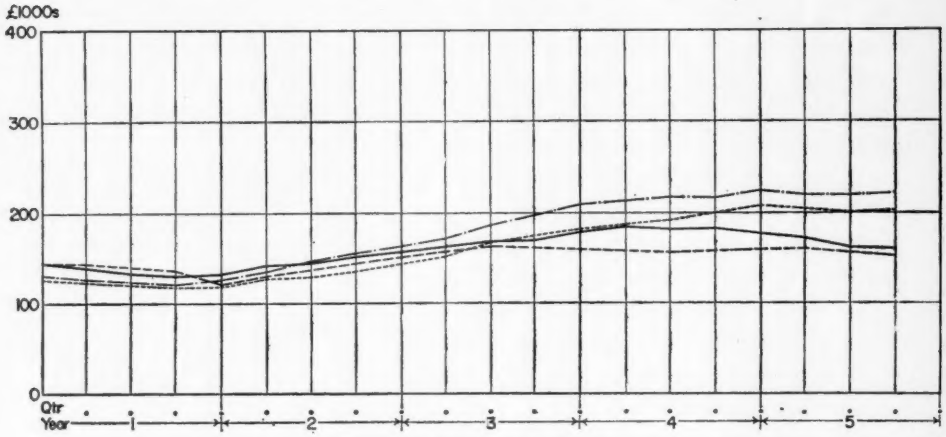

FIGURE 5. *New game: assets.*

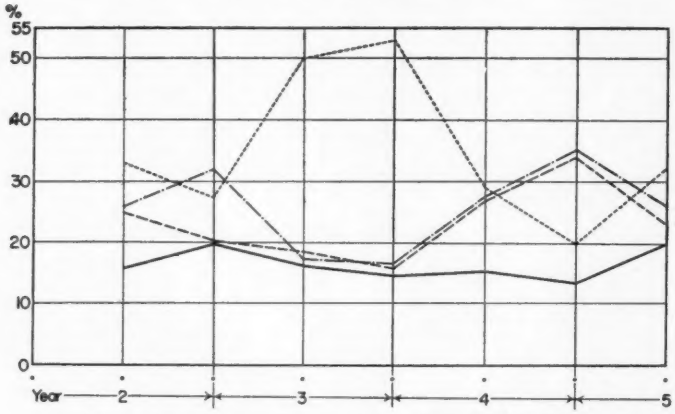

FIGURE 6. *Andlinger game: market share (half yearly).*

4. *New Model*

The information given in this section is known to the umpires only.

(a) *Sales Potentials*

An initial potential of 296 units is distributed between the twenty-four areas so that:

2	have	zero	potential	
11	,,	a potential of	8	units
8	,,	,,	,, 16	,,
2	,,	,,	,, 24	,,
1	has	,,	,, 32	,,

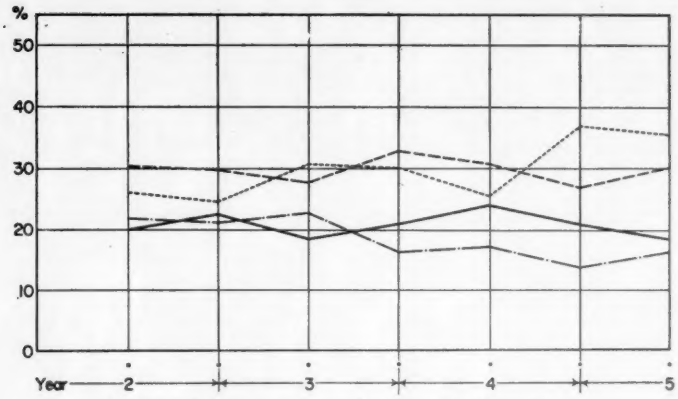

FIGURE 7. *New game: market share (half yearly).*

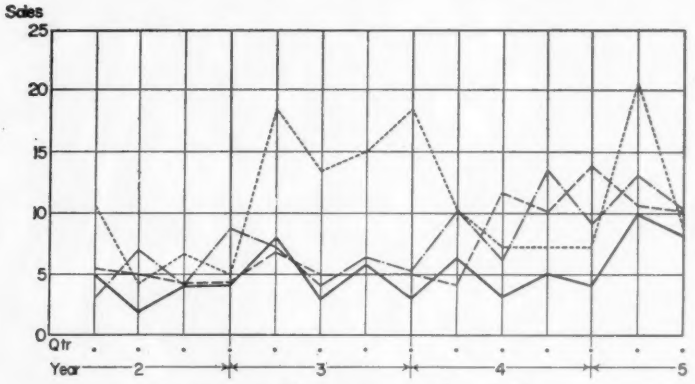

FIGURE 8. *Andlinger game: number of sales.*

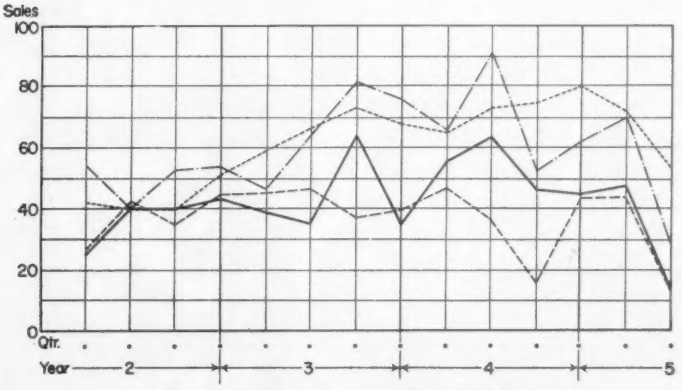

FIGURE 9. *New game: number of sales.*

The allocation is made at random within the four regions, control being exercised only to ensure that there is not a wide difference between regional potentials.

In the early stages of the game, when the market is a rising one, the total potential is increased each quarter by raising the individual potentials of two or three areas by a further 8 units. Areas are chosen at random from those regions that have received the most advertising in the preceding quarter.

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(b) *Rules for deciding Sales*

(i) Every salesman calling on an area gets 25 per cent of the potential in that area; two salesmen from one company obtain a combined share of 50 per cent, three get 75 per cent and four or more will get the full potential for the area, subject to rule (v) below.

(ii) Advertising in a region gives a bonus for each area called on in the region, of sales as follows:

1 Quota of advertising	1 unit
2 or 3 Quotas	2 units
4 or 5 Quotas	3 „

(iii) An improved product gets a bonus of 2 units in all areas visited.

(iv) Market research (£8,000) carries bonus of 2 units for each area visited in region concerned, for current quarter.

(v) If only one company calls in an area, a further allocation is made to it of 50 per cent of the balance of the potential after its basic allotment has been made.

Examples

One salesman from one company calls in an area with potential of 16 units. He gets 4 as his 25 per cent share and a further 6 representing half the remaining potential (after the basic 4 have been allotted), a total of 10.

Two salesmen from one company with an improved product call in an area of potential 24 where 2 quotas of advertising are effective. The allocation to the company is made up as follows:

50 per cent of potential	12
Improved product	2
Advertising	2
	16
Half residual potential	4
Allocation	20

(vi) If two or more companies call in an area, the first to get sales will be that which has the advantage in:

Improved product
Number of salesmen
Advertising
Market research (if in current quarter)

in this order.

This company is also allocated 50 per cent of any remaining potential after all orders have been placed with the companies concerned.

(c) *Manufacturing Costs*

Initial plant now makes up to 50 units per quarter and extra plant can be bought to make a further 50. Production may be varied but only in lots of 10 units.

Costs per quarter are as follows:

Initial Plant	*Initial plus Secondary*
Fixed £3,000	Fixed £6,000
Variable £100 per unit	Variable £90 per unit

Players know the costs for initial plant; they can pay to find the costs relative to secondary plant, in advance, as in para. 2(g).

(d) *Research and Development*

A company may invest in research and development by allocating money in units of £4,000 so that a fund is gradually accumulated. According to the size of this fund there is a probability of achieving an improved product, given in the following table:

Cumulative Amount (£)	*Probability* (*per cent*)
4,000	2
8,000	4
12,000	8
16,000	16
20,000	25
24,000	33
28,000 and over	50

Thus at the end of each quarter the umpire notes the total amount of money his company has invested in research and development, finds the equivalent probability and checks the random number table to see whether they have achieved an improved product.

(e) A summary of income and expense items is given in Table 1.

TABLE 1

SUMMARY OF INCOME AND EXPENSE ITEMS
(Initial capital £160,000)

	£
Selling price per unit	400
Salesmen	
(a) Cost to engage	4,000
(b) Salary per quarter, whether in field or training	400
Advertising	
One quota of advertising (covering one region) per quarter (with maximum of 5 quotas per quarter per region)	1,200
Plant	
(a) Initial plant (maximum production of 50 units/quarter)	60,000
(b) Each additional production line (maximum 50 units/quarter)	12,000
Research and Development	
Minimum per quarter (no upper limit)	4,000
Production (at 50 units of production in batches of 10)	
(a) Fixed cost per quarter	3,000
(b) Variable cost per unit per quarter	100
Staff Work (information)	
(a) Total market potential	800
(b) Share of market	800
(c) Total industry research and development expense	400
(d) Production costs in advance	800
(e) Market research for one region	8,000
(f) Follow up survey after buying (e)	800
Penalties	
(a) Factoring of accounts receivable	20%
(b) Missing a deadline set by umpires	No sales

5. *Features of the Game*

(a) To run the game it is advisable to have a chief umpire and then one umpire for each company. At a deadline the procedure is as follows:

(1) Umpires collect Decision Forms and meet together.

(2) Senior umpire, who maintains list of current sales potentials, runs through areas; as each number is reached umpires call out details if concerned, i.e. whether improved product, number of salesmen, amount of advertising, and sales are allocated and noted on Decision Form.

(3) Senior umpire returns one copy to teams.

(4) Umpires complete Quarterly Summaries.

(5) Umpires check random number tables for (a) loss of salesmen, (b) improved product.

(6) Umpires rejoin teams, return Quarterly Summaries, and start to prepare next Summary as far as they are able. Steps (1)–(3) take less than 5 min, steps (4)–(6) about 5 min.

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(b) It will be seen from the rules for allocating sales (Section 4(b)) that competition plays an important part in the game. For example a team which operates without meeting any competition can make good sales without spending money on advertising or in getting an improved product, but once more than one company is involved this is no longer true. If a company's share of the market is to be maintained it then becomes essential for it to find some way of increasing its sales, and interesting competitive situations develop. The senior umpire can always control the exercise by suitably adjusting market potentials, and it is usual to have a recession towards the end when teams are fully conversant with the game.

(c) A member of each team is nominated at the beginning of the exercise to report at the end to the whole course the philosophy with which his team began, how it was developed in the course of the game and how it could have been improved. The senior umpire maintains blackboard graphs of "total assets" and "share of market" for each company and can use these to bring out any points during the reporting back.

(d) The game has been played under the title "An Exercise in Business Management" at study groups held for middle management and their immediate subordinates, and the response has invariably been good. The normal practice is to have a briefing session in the morning of some 30 min (booklets having been issued beforehand) followed by another 40 min in which each umpire sits in with his team. The game is then played all day, with a 45 min review session at the end; some 5–6 years trading is covered in this time.

It provides a stimulating exercise in taking decisions, and in the use of delegation to permit the information required for such decisions to be made available in the brief time permitted for each quarter. It brings out the need for co-ordination of effort, and the need to obtain information.

6. *Future Developments*

Apart from developments in the rules of the game, to accord with the purpose for which the exercise is being carried out, there is obviously a great deal of more fundamental work that can be done. In the first place we have a situation in which groups of four people are being faced with the necessity to make decisions under pressure in the light of changing circumstances, and we should make more use of this to study how the decisions are taken, and the speed with which a team reacts, say, to a

recession. As it stands the exercise is not suitable for the purposes of personnel selection, but it might be developed to be so.

If the game were incorporated in a management course of several weeks' duration, it would be possible to build case studies into the structure so that, for example, companies could be faced with a situation in which a strike would occur unless the right decisions were taken—and failure could be penalized by holding up production for a period.

Umpires might also take a more analytical role in such a course and it would be valuable to co-opt instructors in appropriate subjects occasionally to sit in with the groups.

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ACKNOWLEDGEMENTS

Obviously a great debt is owed to the original publication[1] which gives a comprehensive account of the Andlinger model. The game described above has been developed largely by incorporating suggestions made by members of the Training Section and the Organization and Methods Dept. of Shell-Mex and B.P. Ltd. who have undertaken the umpiring role, and from those of the company who have been participants in the exercise. The author is grateful to Shell-Mex and B.P. Ltd. for permission to publish this paper.

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A Queueing Model for Spare Coal Faces

F. J. TOFT and H. BOOTHROYD
National Coal Board

Determining the best provision of spare coal faces to allow for unexpected closures of faces can be considered as a problem in queueing theory. A machine-interference type of model is examined with random input, exponential service time and a fixed maximum number of spares. Steady state results are derived for the probability of a given number of faces being out of action, for the average number of spare faces held and for the average loss of working time. Numerical values of the average loss of working time have been computed as a guide to management and examples are given. No practical applications have yet been made.

COAL faces may be subjected to unforeseen temporary or permanent closure. This may be due to geological faults or other causes such as roof falls. If production is to be maintained when this occurs, spare faces have to be kept which can go into production almost immediately as the need arises. In an area where this unforeseen closure occurs frequently a serious problem is to determine the number of spare faces and the rates of opening new faces which are required to keep the loss in coal production to an acceptable level.

The problem can be regarded as a queueing problem and in this paper it is treated in a way very similar to a machine interference model. A number w of working faces is needed for the desired output of coal. A maximum number s of spare faces is held available. Immediately one of the working faces closes, a spare face (if available) is put into production. As soon as the total number of faces available for working drops below $w+s$ a repair team starts to repair or replace a face. The number of repair teams at work is equal to the number of faces short, up to a maximum of c teams; if the number short exceeds c, the remainder form a queue. The working time of a face—from going into production to its next closure—has a probability distribution $a(t)$ and the time for repair or replacement has a distribution $b(t)$.

This model implies a number of simplifications of the system it represents. It ignores any delay in starting production or repair work on a face. All faces are treated as equivalent although in practice there would be differences in possible outputs and in costs of production. The number of repair teams available is fixed at c although more men would become available for repair work when the number of faces working dropped

below w. It differs from the usual machine interference models by providing for some units to be held as spares.

A further simplification is needed to make the problem easy to solve analytically. This is to suppose that the distributions $a(t)$ and $b(t)$ are both exponential and the same for all faces. We can then write:

(i) the probability that a face which is working at time t will close in the interval t to $t+dt$ is λdt.

(ii) the probability that a face being repaired at time t will become available for working in the interval t to $t+dt$ is μdt.

A partial solution for this case has been given by Taylor and Jackson in 1954.[1] The first step in solving the problem is to calculate the probability of any given state of the model, in terms of the variables which specify the operating conditions. It is then, in principle, possible to work out how the costs are affected by the operating conditions.

Let P_n be the probability that there are n faces short (awaiting repair or being repaired) at time t,

and as above,

w = required number of working faces,

s = maximum number of spare faces,

c = number of repair teams,

$1/\lambda$ = average working life of a face,

$1/\mu$ = average time to repair a face.

Then following the method used by Erlang it can readily be seen that:

for $s \geqslant c$:

$$\left.\begin{aligned}
\frac{dP_n}{dt} &= w\lambda P_{n-1} - (w\lambda + n\mu) P_n + (n+1)\,\mu P_{n+1} && (0 \leqslant n < c) \\
&= w\lambda P_{n-1} - (w\lambda + c\mu) P_n + c\mu P_{n+1} && (c \leqslant n \leqslant s) \\
&= (w+s-n+1)\,\lambda P_{n-1} - [(w+s-n)\,\lambda + c\mu]\, P_n + c\mu P_{n+1} && (s < n \leqslant w+s)
\end{aligned}\right\}$$

for $s < c$:

$$\left.\begin{aligned}
\frac{dP_n}{dt} &= w\lambda P_{n-1} - (w\lambda + n\mu) P_n + (n+1)\,\mu P_{n+1} && (0 \leqslant n \leqslant s) \\
&= (w+s-n+1)\,\lambda P_{n-1} - [(w+s-n)\,\lambda + n\mu]\, P_n + (n+1)\,\mu P_{n+1} && (s < n < c) \\
&= (w+s-n+1)\,\lambda P_{n-1} - [(w+s-n)\,\lambda + c\mu]\, P_n + c\mu P_{n+1} && (c \leqslant n \leqslant w+s)
\end{aligned}\right\} \quad (1)$$

(where $P_{-1} = P_{w+s+1} = 0$).

The steady state solution can be found by putting these rates of change equal to zero. The steady state probability of n faces short will be denoted by p_n and the ratio of the average repair time to working time by ρ. (i.e. $P_n \to p_n$ as $t \to \infty$; $\rho = \lambda/\mu$.)
The resulting equations give:

$$\left.\begin{array}{lll}
\text{for } s \geqslant c: & p_n = w^n \rho^n p_0/n! & (n \leqslant c) \\
& = w^n \rho^n p_0/c!\, c^{n-c} & (c \leqslant n \leqslant s+1) \\
& = w!\, w^s \rho^n p_0/(w+s-n)!\, c!\, c^{n-c} & (s \leqslant n \leqslant w+s) \\
\text{for } s < c: & p_n = w^n \rho^n p_0/n! & (n \leqslant s+1) \\
& = w!\, w^s \rho^n p_0/n!\,(w+s-n)! & (s \leqslant n \leqslant c) \\
& = w!\, w^s \rho^n p_0/c!\, c^{n-c}(w+s-n)! & (c \leqslant n \leqslant w+s)
\end{array}\right\} \quad (2)$$

For both conditions, p_0 is given by

$$\sum_{n=0}^{w+s} p_n = 1. \quad (3)$$

To decide the optimum operating conditions two costs need to be considered: the cost of keeping open the spare faces and the cost of losing production time on the working faces. The cost of the repair teams is not affected; the same amount of repair work is always needed for a given amount of coal and the teams are usefully employed elsewhere in the colliery when not working on the faces. The average number of spare faces held at one time is

$$\sum_{n=0}^{s} (s-n) p_n \quad (4)$$

and the average proportion of working time lost (as a result of n exceeding s) is

$$\frac{1}{w} \sum_{n=s+1}^{w+s} (n-s) p_n. \quad (5)$$

These quantities can be calculated using the results above; if the costs per day of a spare face and of losing a working face are known the optimum values of s and c can be found. Unfortunately these costs are difficult to determine and differ widely from face to face. It has not been possible to give a general answer to the question of what spare face provision should be made. The best compromise has seemed to be to provide graphs showing the proportion of working time lost in various conditions. The colliery manager can then assess at any time whether further expenditure on spare faces is likely to be justified by the expected saving in working time.

Specimens of the graphs are shown in Figures 1 and 2. The proportion of time lost is plotted against ρ for various small values of w, s and c.

As an example we may take a colliery which normally has six working faces. Suppose it is known that a face lasts on average 10 weeks before

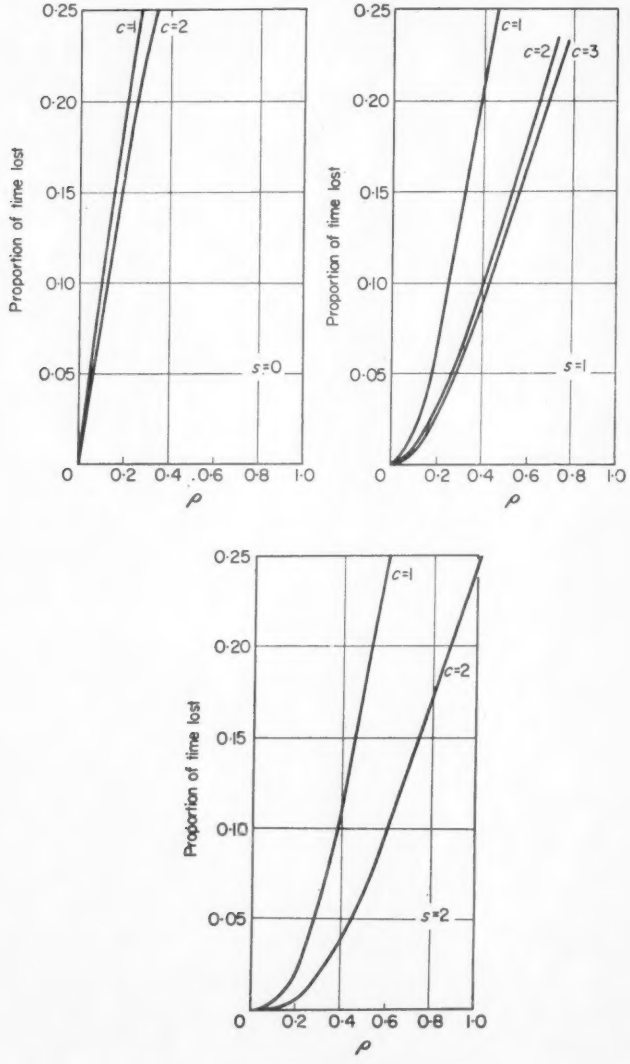

FIGURE 1. *Time lost for two working faces.*

an unforeseen stoppage occurs, and the average time for repair or replacement is 2 weeks. This gives $\rho = 0{\cdot}2$. Then the manager may consider holding various numbers of spare faces and repair teams and the graphs or the formula would give the following estimates of loss of working time.

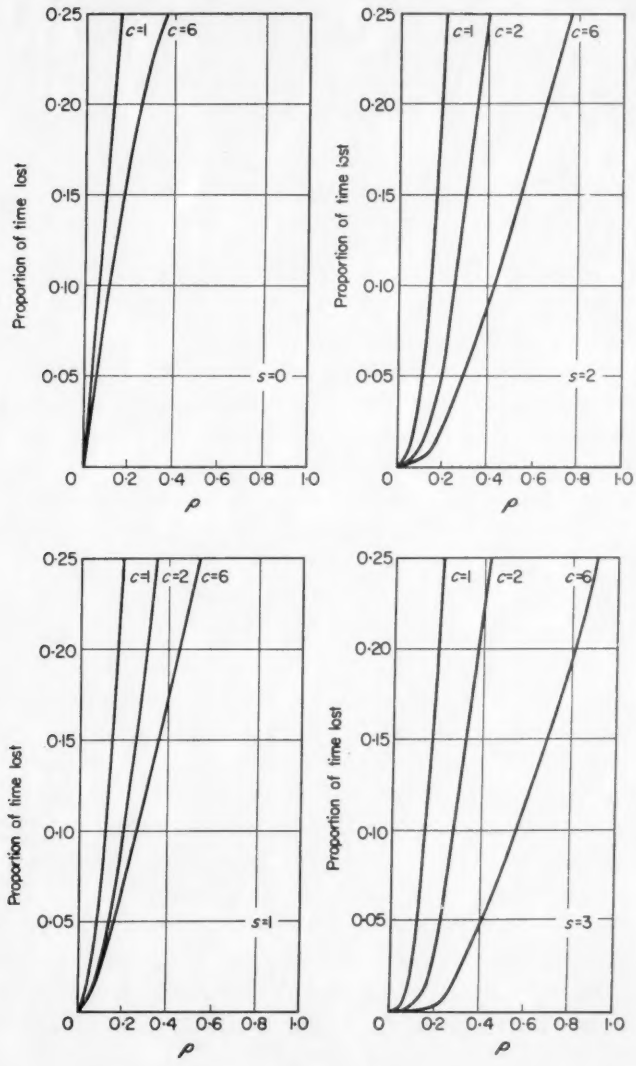

FIGURE 2. *Time lost for six working faces.*

It can be seen that there is little advantage in having more than four repair teams, but up to three spare faces may be desirable, depending on the relative costs of providing the spares and of losing output.

TABLE 1

PERCENTAGE LOSS OF WORKING TIME

No. of Spare Faces	0	1	2	3
No. of Repair Teams:				
1	33	25	23	20
2	18	10	5	3
4	17	7	2	½
6	16	7	2	½

We have not yet been able to check how closely the model fits the facts. It is expected to be reasonably accurate if restricted to unforeseen temporary closures of faces. The occurrence of such closures is probably effectively random. The distribution of repair times is more doubtful. It will range from almost zero up to the time needed to open a new face, with steadily decreasing frequency, but we do not know how close it is to exponential. For faces which close permanently, the service time is the time to open a new face. This time is much less variable than the repair time and its distribution is certainly not exponential. The model we have studied will not, therefore, apply if permanent closures are to be considered. The greater regularity of service time will lead to a smaller loss of working time for a given value of λ. Constant service time may be taken as an extreme case; Ashcroft[2] has given results when there are no spares. Over the range we have considered his figures are very close to those for the exponential model. The introduction of spares increases the effect of the service time distribution.

In this paper we have not answered the original question—what is the optimum provision of spare coal faces at a colliery? Instead we have provided an estimate of the loss of working time (or an upper limit of it) as a guide to management in considering the question. Steady state results have been used but in practice circumstances are not likely to remain constant long enough for equilibrium to be reached. The results should be interpreted as indicating the direction of the trend, rather than the actual losses to be expected at any time. A guide of this sort is probably as much as is desirable in the present context; a model which took into account all the relevant peculiarities of a colliery would be impossibly complicated.

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ACKNOWLEDGEMENTS

The authors would like to thank Mr. D. Taylor, lately a member of the National Coal Board, for his help in preparing this paper. The paper is published by permission of the National Coal Board.

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Letter to the Editor

Is Your Fleet the Right Size?

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THE following result is simple, and yet possesses such a range of application that wider publicity seemed desirable.

The problem arose during studies of small railway systems. One of the questions to be answered was "Is the wagon fleet the right size?" If the fleet is too large the wagon utilization will be low, whereas if the fleet is too small wagons need to be hired at extra cost.

A simple rule to assess whether the fleet is the right size is:

"If hired wagons are k times as expensive as owned wagons, then wagons should need to be hired on 1 day in k days."

The proof is as follows:

Suppose the fleet consists of n wagons, each costing 1 unit per day, hired wagons costing k units per day, and suppose that on any day x wagons are required with probability $p(x)$.

Then the total expected cost is given by C, where

$$C = n + k\int_n^\infty (x-n)\, p(x)\, dx$$

and if n has been chosen to minimize C

$$\frac{dC}{dn} = 0$$

i.e. $$1 - k(n-n)\, p(n) - k\int_n^\infty p(x)\, dx = 0$$

i.e. $$\int_n^\infty p(x)\, dx = \frac{1}{k}$$

which means that the fleet should be adequate except for a proportion $1/k$ of the time.

If for wagons you read lorries, or men, the wide application of this result becomes apparent.

DAVID KIRBY

National Coal Board,
Field Investigation Group,
18–20, Albert Embankment,
London, S.E.11.

Abstracts and Reviews

Portfolio Selection: Efficient Diversification of Investments.
HARRY M. MARKOWITZ.
John Wiley and Sons, New York; Chapman and Hall, London, 1959. 344 *pp.* 60*s.*

After a 30-page introductory section, the next 90 pages (Part II) of this book consist of a pleasant and easy-going introduction to the ideas of probability and statistics, which the author invites the mathematically-trained reader to skip or skim. Part III is concerned with the selection of "efficient" (perhaps better called "admissible") portfolios of securities. A portfolio is called efficient if its expected return cannot be increased without increasing its variability of return and if its variability cannot be reduced without also reducing its expected return. The statistical computations required are tedious and, for a large number of securities, necessarily electronic, consisting of a preliminary calculation of historical average returns, and variances and covariances of returns, of all securities considered, followed by a linear programming routine to find efficient combinations of them. The matrix methods necessary for the statement of the programming problem, and its solution in simple cases, are developed in the text, but I doubt whether matrix algebra can be taught in one chapter, and consequently I doubt whether anyone without matrix method experience will penetrate Part III with full understanding.

Part IV contains an interesting 75-page exposition of the modern utility theory of choice (which would be a good homeopathic introduction to a more profound discussion, such as that in Savage's *Foundation of Statistics*) followed by 30 pages of what seemed to me rather sanguine discussion of the possibilities of applying this theory to portfolio analysis.

It will be seen that the subject-matter of the book is (like many portfolios) over-diversified, as is to be expected of a reincarnated Ph.D. thesis. The style, however, suffers from none of the customary turgidity. Part III, in particular, would be of interest to the operational research linear programmer.

I suppose that "efficient" portfolios in this book's sense are what large investors often seek, but I object to them on two grounds. In the choice of securities, historical data are not nearly a sufficient guide: today's

fashions in securities must be shunned in favour of what we think are tomorrow's. (Throgmorton Street is haunted by former True Blue Chips which are now alone and palely loitering, and to avoid a due proportion of these in a portfolio is an achievement.) My second objection is related, but more radical: given the list of securities to be bought, the efficient timing of entries into the market has an effect on portfolio performance which dwarfs that of the more conventional calculations of yield. How does one achieve efficient timing? In the present state of knowledge, by examining the past record of the security, by patience, and by "hunch" based on experience. This is a problem which would amply repay *unpublished* statistical investigation!

ALAN STUART

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Digital Computing Systems.
SAMUEL B. WILLIAMS
McGraw-Hill, New York, 1959. 224 *pp.* 60*s.*

Electronic digital computers are being used more and more for the processing of complex large-scale operational research studies, and therefore some knowledge of computers would seem to be essential to the competent operational research investigator.

In the preface the author states that the purpose of his book is to describe the structure of electronic computing systems in a manner understandable to those who are familiar with electrical circuits and apparatus but who have little or no knowledge of mathematics. Since operational research workers are presumably more familiar with mathematics than electrical circuits this book may not tell them what they need to know about "digital computing systems".

The book is not heavy going, however, and should be easily followed by a conscientious reader. The first half contains details of existing input, output, printing and storage systems for computers. Unfortunately only American machines are described and well illustrated, and in the research and development of computers no credit is given to work done in the U.K. after Babbage in the 19th century, except for the cathode ray tube memory system of Professor F. C. Williams.

Most of the second half of the book is concerned with basic circuits employing electro-magnetic relays, vacuum tubes, transistors and magnetic transformers, and there are many diagrams of flip-flops, gates and adders. But the exposition is hardly done with the thoroughness or authority one would expect from a book devoted to the subject of circuits. Final

chapters on programming and on applications are useful but can only introduce the subjects in the space available. A little is said about numerical analysis preceding the programming, but these chapters would not give an insight to the operational research worker on how to consider using a computer for his problems.

Digital Computing Systems is one book in the McGraw-Hill series in "Information Processing and Computers", and the needs of operational research specialists are better catered for in others of the series, such as: Gotlieb and Hume, *High-speed Data Processing*, or Davis, *Computability and Unsolvability*.

R. M. PAINE

Quality Control and Industrial Statistics (Revised edition).
ACHESON J. DUNCAN.
Richard D. Irwin, Homewood, Ill., 1959. xxxi + 946 *pp.* $9.00.

This is a sound textbook of the main statistical methods used by engineers (in the American sense of the word) in industry. It is too long and all-inclusive to be read as an exposition—the mind quails at the thought of anyone being asked to absorb its eight or nine hundred pages. But it would form a good accompaniment to a course of lectures or study, and provides an excellent reference book. Very little that has been done on the subjects covered seems to have been missed, and the list of references is formidable. There are also copious appendices giving mathematical proofs.

It is doubtful, however, if this is the book on statistics that operational research workers are looking for. Nearly three hundred pages are devoted to describing sampling inspection schemes and control charts as used in quality control. All this has long since left the sphere of the research worker for that of the practitioner. Operational research workers sometimes have to analyse data with the aid of methods and significance tests based on frequency distributions, the χ^2 test, the analysis of variance and co-variance; and if they need to be reminded of these methods they will find them well displayed in this book. But operational research workers are sophisticated people who are more likely to be interested in such subjects as the design of experiments and the mapping of response surfaces, which is dealt with only in an introductory way in about seventy pages, or in the ramifications of queueing theory or stochastic processes, which are not dealt with at all.

L. H. C. TIPPETT

Analyses of Industrial Operations.

E. H. BOWMAN and R. B. FETTER (Editors).

Richard D. Irwin, Illinois, 1959. viii+485 *pp.* $7.95.

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This book is a collection of the papers, culled from many journals, used by the authors for several years as illustrative examples in the graduate course in M.I.T.'s School of Industrial Management. They are all intended to be examples of the use of techniques rather than descriptions of the techniques themselves and thus are meant for someone with a considerable understanding of operational research methods.

The techniques considered are linear programming; inventory; stock and production control; and queueing theory. The methods of solution are mainly mathematical. No simulation studies are included and no operational research techniques other than those mentioned are considered. The book is therefore somewhat restricted in scope and if used for teaching purposes would only cover part of an adequate operational research curriculum. Its main value, however, lies in the thoroughness with which the applications of the techniques are explored and how, in almost every paper, the problems of using the techniques within a given situation are discussed.

In the linear programming section for example, the following uses of linear programming are discussed: the oil industry (blending gasoline, balanced oil well production, refinery operation, transportation problems); maximizing profit of a machining unit; switching of trucks in American freight cars; minimizing production costs of electricity; routing of aircraft; metal mixing; warehousing and distributing a seasonal product; and optional scheduling of a production unit.

It is useful to have these papers in a group and to be able to study the different methods of formulating the problems into a linear programming framework and to compare the approximations and simplifications which are made in order to use the analytical approach. The impression is conveyed with some of the papers that the imposition of the linear programming model is a little forced.

The two most valuable papers of this section are one by Jewell in which the transportation matrix is applied to the production scheduling of a seasonal product—this is a down-to-earth paper and makes good reading —and one by Salveson on optimal machine loading in which the simple "square root" formula for ordering of batches is shown to be inapplicable where many products are made in the same plant.

Part II of the book begins with the two classical papers by Holt and his colleagues on their linear decision rule for production scheduling and includes papers on production control in the British Dunlop Rubber Co. and an American chemical firm. The contrast in scope between these two adjacent papers is very striking. The section ends with a description of a production scheduling scheme for component assembly in the I.B.M. company calculated on an I.B.M. 701 computer. This paper is disappointingly short, considering that it is one of the few published applications of the technique.

In section III the queueing applications discussed include the problem of an operator attending several automatic machines. This paper by Mangelsdorf is certainly one of the best in the book, probably because it is part of a S.M. Thesis at M.I.T. As, to my knowledge, it has not been published elsewhere, it is worth borrowing the book for this paper alone. It suffers a little for being somewhat academic but the many graphs given should provide food for thought for anyone facing this type of problem. The other queueing applications include Brigham's aircraft factory case, one of Eddison's steelworks examples and an American warehousing problem.

The final two sections, although given distinctive headings, do not really consider a different class of techniques or problems from the earlier ones. The emphasis is a little more on economic criteria of effectiveness. Some interesting cost functions emerge, but their relevance to particular situations can only be determined by those actually in the situation. The most interesting paper in this part of the book is that by Vernon Smith on the economics of replacing equipment.

To sum up, the book is a worthwhile collection of attempts which have been made to subject industrial operations to mathematical analysis. I use the word "attempts" because not many of the papers indicate that the results of the analysis have been put to practical use. This is a disappointing feature of the book but one which is, alas, all too common in operational research literature.

For an American book the editors have been generous in including three papers (out of twenty-six) from British sources. It is a solemn thought, however, that no papers were chosen from the *Operational Research Quarterly*.

W. E. DUCKWORTH

High Speed Computing. Methods and Applications.

S. H. HOLLINGDALE.

English Universities Press, London, 1959. 244 *pp.* 25*s.*

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As described in the foreword, this book will be of value to the research worker who wishes to assess the potentialities of digital computers. The general tone is set on this note, so that, while written in an easy and readable style, no attempt is made to gloss over the more technical aspects of computers and programming.

In the first four chapters, computers and programming are introduced. The reader who is unfamiliar with mathematical symbolism may well falter here, but should persevere with the thought that many more words would be required to describe the same material.

In particular the chapters on representation of numbers and an introduction to programming are comprehensive in their material, yet do not occupy a large part of the book. The short historical chapter which follows provides some relief and covers the early computing machines which paved the way for the computers of today.

The following two chapters are devoted to a detailed description of two British Computers, EDSAC I and DEUCE. The author has chosen the DEUCE because of his great personal experience with this computer, and is to be congratulated in compressing so comprehensive a description of programming this far from simple machine into fewer than 50 pages. A complete understanding of these chapters should satisfy the reader that he could absorb without difficulty the logic of any other computers on the market today.

The following two chapters on storage and logical design fill in some of the gaps left in the layman's knowledge of computers. In these, the accuracy of text and diagrams continue to do much to make a fairly advanced discourse clear.

The remainder of the book covers such varied subjects as computing services, applications mainly of a scientific nature, process control and language translation. While not possessing the continuity of idea contained earlier in the book, this part does give some idea of the versatility of electronic computers.

The book is to be recommended to the serious student of computing, but parts of it may delve too deeply into the subject for the fire-side reader.

M. C. HANCOCK

Operational Research Case Studies.
JOHN HARLING.
J. Instn Prod. Engrs, **37**, 644–649; discussion 650–652 (Oct. 1958).

A discussion of the definition of operational research is used to introduce typical examples classified as: control of stocks and production scheduling; programming and allocation problems; waiting line and congestion problems and miscellaneous problems such as bidding by tender and price structure determination. These are discussed in general terms and types of problem with only indications of procedures and solutions are given. A discussion of relations to related topics—cost accounting, work study, econometrics and computer programming is added.

Cybernetics, Operational Research and Automation.
F. H. GEORGE.
J. Instn Prod. Engrs, **37**, 634–643 (Oct. 1958).

A discussion is presented of the nature of cybernetics and operational research and their relation to automation ranging over scientific method, probability and statistics, language and logic and via the theory of games to cybernetics.

Operational Research in Industry.
W. E. DUCKWORTH.
Metal Treatm., **25**, 369–375 (Sept. 1958); 417–423 (Oct. 1958).

This article was one of two received in response to the editorial note of November 1957 reported at the time in the *Operational Research Quarterly*. It sets out the sort of problem that operational research can solve by the eight ordinary processes (statistics, linear programming, queueing theory, Monte Carlo methods, stock control models, decision functions and game theory and cybernetics). Examples are worked out in some cases. Extensions of operational research beyond the eight techniques mentioned are briefly indicated. (20 references.)

Operations Research in a Metal Rolling Mill.
A. P. M. PURDON.
Metal Treatm., **25**, 503–507 (Dec. 1958).

The second article brought by the editorial comment discusses an investigation of a copper rolling mill. The problem of how it could be improved either by additional construction or by better operation was tackled in a

preliminary period of assembly of the team; in a 6-months period of collection of data; another stage in which a numerical model was constructed and different levels and patterns of production were worked out; finally results and suggestions were presented and discussed. These stages are reviewed in turn, the first in terms of human relations within the team and their work; the sort of data collected is given for the second part and a flow-sheet with comments for the third. The presentation of the results is reviewed, finally, without details or actual figures but the kinds of conclusions arrived at are indicated. In conclusion, the time taken and cost of studies on this scale are considered.

Operational Research in Advertising.
J. MAITLAND.
Advanc. Sci., Lond., **15** (61), 459–467 (June 1959).

The article discusses the method of analogy and applies to advertising data the analogies drawn from physiology, pharmacology and physics. A development of the paper on the same subject was given by Benjamin and Maitland, *Operat. Res. Quart.*, **9** (3), 207–217 (1958).

O.R. and the Accountant.
KENNETH S. MOST.
Accountancy, **70** (793), 460–463 (September 1959).

The article is a general description of the subject of operational research written for accountants. It covers briefly the history of operational research, the approach operational research workers use to tackle problems and cites examples of problems which have been successfully tackled. It goes on to discuss the relationship of the accountant to operational research and to suggest problems which accountants have by them continually and which may prove amenable to operational research techniques.

The article does contain a few common heresies.

"The techniques used (in operational research) are mainly linear and non-linear programming."

"The chief accountant executive of an organisation should decide which problems are suitable for treatment by operational research."

News and Notes

Operational Research Society

Annual Report for the Year Ended 31 July, 1959

AT the Annual General Meeting in October, 1958, the President, Professor M. G. Kendall, introduced the newly appointed full-time Secretary, Miss G. M. Heselton. The results of the elections were announced: Mr. B. H. P. Rivett, Mr. E. C. Williams and Mr. B. D. Hankin had been returned unopposed as Honorary Secretary, Honorary Treasurer and Honorary Membership Secretary of the Society respectively, and Mr. S. L. Cook had been elected to the vacant seat on the Committee (Mr. G. Norton, vice Dr. H. F. Rance, retiring under the four-year rule). The meeting also agreed the recommendation of the Honorary Treasurer that Messrs. Knox, Cropper & Co. should be invited to become the Society's auditors.

At 31 July, 1959, the membership was 524: 267 full, 257 associate.

The chief administrative work carried out during the year has been the establishment of permanent offices, equipping them, and the amalgamation of various sets of records which, in the past, had been maintained by the honorary officers. The Secretary was enjoined to obtain offices at a low rental and after a few months' delay and a great deal of viewing she obtained the pleasant and light offices at 64 Cannon Street, which members of the Society are invited to visit at any time.

The Society, in December, 1958, was approved by the Commissioners of Inland Revenue for the purposes of Section 16, Finance Act, 1958, whereby in certain circumstances subscriptions to the Society may be claimed for income tax purposes as a legitimate professional expense.

The Society was fortunate to receive a generous response from industry to its appeal for funds; a list of our supporters is appended to this report.

The first meeting of the Society's winter programme was the Annual General Meeting held on 23 October, 1958, after which the President gave the first presidential address of the Society: his subject was "The Teaching of Operational Research"; this address was subsequently published in the

December, 1958, issue of the *Quarterly*. The rest of the winter programme was:

	1958	
17 November	*Speaker:*	Mr. Stafford Beer (The United Steel Companies Ltd.).
	Subject:	"What has Cybernetics to do with Operational Research?"
8 December	*Speaker:*	Mr. B. H. P. Rivett (National Coal Board).
	Subject:	"Competitive Bidding."
	1959	
26 January	*Speaker:*	Mr. S. L. Cook (Richard Thomas & Baldwins Ltd.).
	Subject:	"Operational Research and Industrial Engineering—Co-existence or Dynamic Co-operation."
17 February	*Speaker:*	Mr. E. C. Williams (Ministry of Defence).
	Subject:	"Trends in Military Operational Research in the United Kingdom."
11 March	*Speaker:*	Mr. H. R. W. Watkins (Shell-Mex and B.P. Ltd.).
	Subject:	"Business Games in Business."
6 April	*Speaker:*	Mr. J. Stringer (Central Electricity Generating Board).
	Subject:	"Operational Research in Electricity Generation."

All these meetings, except the January one, were held in the Council Room of the Agricultural Research Council, which was made available to the Society, as it had been in past years, by the kind permission and generosity of Sir William Slater. When planning the programme it was felt that the time had come when one meeting at least should be held out of London, and in consequence the January meeting was held at the University of Sheffield, and was preceded by a sherry party given by Dr. J. M. Whittaker, the Vice-Chancellor. The arrangements were handled by Mr. R. S. Gander and our warmest thanks are due to him; there was a record attendance and the experiment was a great success. The future programmes will almost certainly contain at least one meeting per session out of London, and the Committee are planning that in 1959/60 this will be held in Cardiff.

It is felt that the time has arrived when the Society should meet in a larger room and, in consequence, must be prepared to meet the costs.

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The meetings during 1959/60 will be held in the lecture theatre of the Royal Society of Arts. It is also planned by way of experiment that a dining club will be organized for the benefit of those members attending the meeting who wish to stay on in town for a meal and to talk to other members of the Society.

A number of American operational research workers will be visiting Europe during the beginning of September and the opportunity will be taken to hold a meeting during the afternoon of Monday, 14 September, at the Royal Society of Arts and which will be addressed by Professor C. West Churchman, Mr. Georges Brigham and Dr. Herbert Holt; this meeting will be followed by a dinner at the Charing Cross Hotel.

The Society's National Conference took place from 10 to 12 June at the Old Swan Hotel, Harrogate, and was attended by 120 people. The Committee are grateful to Mr. J. Stringer and his Committee for all their work.

A 2-week course on operational research was organized in April in conjunction with the London School of Economics and was adjudged to be a great success; it is hoped this course will be repeated in the future. The Society are assisting the British Institute of Management to organize a 1-day conference in September, 1959, entitled "Operational Research at Work"; this conference will be primarily for management. During the year the Educational Sub-Committee has given assistance to various Colleges and Institutes on courses and lectures.

On 1 January the International Federation of Operational Research Societies (IFORS) officially came into being and this Society is one of the three founding members, the other two being the Societies of America and France (ORSA and SoFRO); the Secretary is Sir Charles Goodeve, O.B.E., F.R.S. This Society's representative on the IFORS Board of Representatives is Mr. B. H. P. Rivett. Work has already begun in connexion with the second international conference on operational research which is being organized by IFORS and which will be held in Aix-en-Provence from 5 to 9 September, 1960. Mr. B. H. P. Rivett is the Chairman of the Programme Advisers Committee.

The *Operational Research Quarterly* is now published by Pergamon Press Limited, as reported in the previous annual report. The circulation continues to increase, and at June, 1959, the number of subscriptions, including Society members, was: U.K. 659, U.S.A. 397, others 435 (total 1,491). The total number of pages in Volume 9 (1958) was 328, which is the greatest number in any year to date. There has been an increase in the number of

papers submitted for publication, which is most encouraging, and it is hoped that this tendency will continue.

Mr. Max Davies resigned as one of the joint editors at the end of 1958 and his resignation was received with great regret. Since his resignation there have been some difficulties in ensuring the publication of the *Quarterly* on due date; it was unfortunate that just when it appeared there was some prospect of overcoming these delays, the printing strike caused complete confusion and the June issue, even in an attenuated form, was still not expected to be published by the end of August.

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The Executive Committee has appointed an editorial committee consisting of:

Mr. R. T. Eddison (*Chairman*)
Mr. R. H. Collcutt
Mr. S. L. Cook
Professor P. M. Morse
Dr. R. J. Smeed
Mr. H. R. W. Watkins

to continue the editorial function. This committee is examining the aims and objects of the journal and is considering how it can be expanded and improved.

The Executive Committee plans to place before the membership for their ratification a new draft of the Constitution embodying changes in the periods of office of the Committee members, and in their number; they are also planning to recommend to the membership that the Society be incorporated as a Company without share capital and limited by guarantee.

Professor Kendall retires as President in October 1959 and the Society are greatly indebted to him for the work he has done during his term of office, a period during which the shape of the Society has been changing; we are, however, fortunately not to lose his services, as he remains on the Committee for another 2 years as a Vice-President, taking the place of Sir Owen Wansbrough-Jones who is retiring. The Society greatly appreciate the outstanding services Sir Owen has given, and the work he has done. At the forthcoming Annual General Meeting the Committee will be recommending the election of the Earl of Halsbury as President for the period 1959–61, and the Society is fortunate that such a distinguished figure has allowed his name to be put forward.

The Committee regret that they are losing the services of Mr. H. G. Jones, who is retiring under the 4-year rule and they are grateful to him for the support he has given. Mr. R. S. Gander's resignation from the Committee was accepted on 5 January, 1959; Mr. J. Stringer was appointed

by the Committee to complete Mr. Gander's term. The Committee have learnt, with regret, that the Honorary Treasurer, Mr. E. C. Williams, after serving in that capacity for 5 years does not feel able to stand for re-election; they are extremely grateful to him for the work he has done.

Signed:

M. G. KENDALL (*President*)
OWEN WANSBROUGH-JONES (*Vice-President*)
WILLIAM SLATER (*Vice-President*)
S. L. COOK
H. G. JONES
K. PENNYCUICK
J. STRINGER
B. D. HANKIN (*Honorary Membership Secretary*)
B. H. P. RIVETT (*Honorary Secretary*)
E. C. WILLIAMS (*Honorary Treasurer*)

64 Cannon Street,
London, E.C.4.

SUB-COMMITTEES OF THE SOCIETY

Membership:	Mr. B. D. Hankin (*Chairman*)
	Dr. T. E. Easterfield
	Mr. H. G. Jones
Editorial:	Mr. R. T. Eddison (*Chairman*)
	Mr. R. H. Collcutt
	Mr. S. L. Cook
	Professor P. M. Morse
	Dr. R. J. Smeed
	Mr. H. R. W. Watkins
Education:	Dr. K. Pennycuick (*Chairman* from 5.1.59)
	(Mr. R. S. Gander resigned as Chairman and member on 5.1.59)
	Mr. S. L. Cook (from 5.1.59)
	Dr. F. G. Foster
	Dr. G. A. Garreau (from 6.7.59)
	(Mr. B. H. P. Rivett (to 6.7.59))
Finance:	Professor M. G. Kendall (Chairman)
	Sir Owen Wansbrough-Jones
	Mr. S. L. Cook
	Mr. B. H. P. Rivett
	Mr. E. C. Williams

ad hoc Sub-Committees:
Harrogate Conference 1958*:*
Mr. J. Stringer (*Chairman*)
Mr. H. R. W. Watkins
(Mr. P. G. Lucas—resigned)

Organization:
Professor M. G. Kendall (*Chairman*)
Mr. R. T. Eddison
Mr. B. H. P. Rivett
Mr. E. C. Williams

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FINANCIAL SUPPORTERS OF THE SOCIETY

British Iron and Steel Research Association
British Oxygen Co. Ltd.
British Petroleum Co. Ltd.
British Tabulating Machine Co. Ltd.
Courtaulds Ltd.
Distillers Co. Ltd.
Electric & Musical Industries Ltd.
Ferranti Ltd.
Arthur Guinness Son & Co. (Park Royal) Ltd.
Pilkington Bros. Ltd.
Rolls-Royce Ltd.
Shell-Mex & B.P. Ltd.
Shell Petroleum Co. Ltd.
Urwick Orr & Partners Ltd.

1959 Annual General Meeting of the Society

THE following are the Minutes of the Annual General Meeting of the Society, held on 21 October, 1959, at 6 p.m. at the Royal Society of Arts, 6 John Adam Street, London, W.C.2: they are subject to confirmation at the 1960 Annual General Meeting.

The Chair was taken by the President, Professor M. G. Kendall.

(1) The Minutes of the Annual General Meeting held on 23 October, 1958, were confirmed: there were no matters arising.

(2) The President presented the Executive Committee's Report for the year ended 31 July, 1959, which had been previously circulated. On summarizing the year's work he stressed the Society's indebtedness to Sir Charles Goodeve and Sir Owen Wansbrough-Jones; the President also

thanked the members of the Executive Committee and the Secretary for the work they had done during the year.

As stated in the Report, the *Operational Research Quarterly* is now published by Pergamon Press Ltd, for the Society. In the light of this arrangement, the Business Editor of the Journal, Mr. Max Davies, had resigned and Professor Kendall thanked Mr. Davies for the very useful services he had given over eight years; this was warmly endorsed by the meeting.

Professor Kendall drew the meeting's attention to the note in the Report that the Executive Committee were proposing to recommend to the membership that the Society be incorporated as a Company limited by guarantee. It was moved that this action be approved in principle: after discussion, motion passed.

The Chairman then moved the adoption of the Report: motion passed.

(3) The Honorary Treasurer, Mr. E. C. Williams, presented the Accounts (previously circulated), and he expressed gratitude to the Honorary Treasurer-elect, Mr. G. Norton, who had been responsible for their production. Mr. Williams went on to say that most of the industrial money is at present promised for three years only and it will be the duty of the Executive Committee in the future to place the Society on a sound economic basis for the longer term. He then moved the adoption of the Accounts: after discussion, motion passed.

(4) The President made the following announcements regarding the elections of the officers:

President: Lord Halsbury was declared elected as President of the Society for the next 2 years.

Honorary Secretary, Honorary Treasurer and Honorary Membership Secretary: Mr. B. H. P. Rivett had been returned unopposed as Honorary Secretary and was declared elected.

Mr. E. C. Williams had resigned as honorary treasurer: the President, on behalf of the Society, thanked Mr. Williams for the valuable services he had given through a long and difficult period. He then declared Mr. G. Norton elected (no further nominations having been received).

Mr. B. D. Hankin had been returned unopposed as Honorary Membership Secretary, and was declared elected.

Executive Committee member: Mr. H. G. Jones' term of office had come to an end and the President took the opportunity to thank him for the work he had put in. Professor Kendall then gave the result of the recent ballot for this vacant place and Mr. Stafford Beer was declared elected.

Professor Kendall said that at this point he would have introduced Lord Halsbury to the meeting, but unfortunately Lord Halsbury, due to unforeseen circumstances, was out of the country and had sent his apologies for absence. The Society was grateful, Professor Kendall continued, that such a distinguished figure had agreed to serve as President for the next two years.

(5) Discussion followed on the ballot procedure, and the Secretary took note of suggestions made by the meeting.

(6) Election of auditors: it was moved that Messrs. Knox, Cropper, Gedge & Co. be re-elected auditors of the Society for the year 1959/60: motion passed.

(7) There being no other business the meeting was declared closed at 7 p.m.

Northern Operational Research Group

We are advised that a Northern Operational Research Group has been formed. The Chairman is Dr. B. V. Bowden, Principal, Manchester College of Science and Technology and the Honorary Secretary Mr. N. W. Fox of Joseph Crosfield & Sons Ltd., Warrington. Committee Members are Mr. Gander, Dr. Knowles, Mr. Stuart and Mr. Tatlow. Meetings are normally held on the third Tuesday in each month, usually at the Manchester College of Science and Technology, at 7 p.m.

News of Members

New Full Members of the Society

A. Battersby—The British Petroleum Co. Ltd., London.
Miss A. G. Doig—London School of Economics, London.
W. D. Ewan—British Nylon Spinners Ltd., Pontypool.
K. B. Haley—Birmingham University.
Dr. B. O. Koopman—U.S. Operations Research Groups, London.
R. L. Lewis—B.I.S.R.A., London.
Professor Shephard—EPA–OEEC, Paris.
E. E. Treadwell—Army Operational Research Group, West Byfleet.

Changes of Appointment

Sir Owen Wansbrough-Jones, K.B.E., C.B.—Formerly Chief Scientist, Ministry of Supply, is now a director of Albright & Wilson Ltd., London.

R. M. Adelson—Previously at Cambridge University is now with B.I.S.R.A., London.

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Georges Brigham—Previously with the Boeing Airplane Company will be affiliated with the Office of Naval Research in Honolulu, Hawaii, for one year.

L. F. Britten—Previously with the Osram Group, is now with the Eastern Electricity Board H.Q., Ipswich.

V. L. Budge—Previously with the United Steel Companies Ltd., is now with the Templeborough Rolling Mills Ltd., Rotherham.

Dr. E. J. Reinger—Previously with the United Steel Companies Ltd., is now with Chrysler International S.A., Geneva.

R. Solt—Previously with T.I. (Group Services) Ltd., is now with the Reed Paper Group, Nr. Maidstone.

R. G. Stansfield—Previously with D.S.I.R., London, is now with the D.S.I.R. Warren Springs Laboratory, Stevenage.

G. Thackray—Previously with the Distillers Co. Ltd., is now with the Shell Chemical Co. Ltd., Carrington Research Laboratory.

G. F. Todd—Previously with The Imperial Tobacco Co., is now a director of Attwood Statistics Ltd., London.

J. Wardley-Smith—Previously with D.S.I.R., London, is now with the D.S.I.R. Warren Springs Laboratory, Stevenage.

P. H. Winter—Previously with Saunders-Roe Ltd., is now with Vero Precision Engineering Ltd., Southampton.

Erratum

In the June issue of the *Quarterly*, under News and Notes, Mr. B. J. Elson was shown as having moved to the British Petroleum Co., Ltd.; this should have been the British Aluminium Co., Ltd.